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COMPUTATIONAL ANALYSIS OF THE NON-NORMAL CYCLIC SUBGROUP GRAPH OF THE QUASIDIHEDRAL GROUP OF ORDER 32 USING GAP AND MAPLE SOFTWARE

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Abstract:

Let G be a group and H be a non-normal cyclic subgroup of G. The non-normal cyclic subgroup graph, denoted as $\Gamma_H^{NN}(G)$, is defined as a directed graph with vertex set elements of G such that for two distinct elements x and y in G, x is the initial vertex and y is the terminal vertex of an edge if their product is in H. In this presentation, the non-normal cyclic subgroup graphs for the quasidihedral group of order 32 are constructed. Firstly, the non-normal subgroups of quasidihedral groups of order 32 are determined using the Groups, Algorithms and Programming (GAP) software where the identification of the non-normal cyclic subgroups is carried out. Subsequently, the definition of

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the non-normal cyclic subgroup graph is applied to compute the directions of the vertices of the graph and Maple 2016 is used for graph visualization. The results show that there are several different graph patterns arise from these subgroups.

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Keyword:

Graph Theory, Group Theory, Quasidihedral Group, Subgroup Graph

Introduction

Graph theory provides a mathematical framework for describing relationships among objects using vertices and edges. It may be classified into several types, including directed and undirected graphs, as well as weighted and unweighted graphs. Many different fields, including science, technology and mathematics, make extensive use of graph theory. Graph theory is actively used in several fields, including biochemistry, chemistry, computer science, particularly in algorithms and computation, communication networks, coding theory and operations research, including scheduling. Moreover, graphs are used in numerous applications, including task scheduling, graph coloring and data mining (Prathik et al., 2016). This study will focus on directed graph which specifically the non-normal cyclic subgroup graph. Subgroup graph was introduced by Anderson et al. (2012), as a simple directed graph. Then, Kakeri and Erfanian (2015), introduced the complement of subgroup graph which only focuses on normal subgroups. Rajkumar and Devi (2016) explored the intersection graph of cyclic subgroups of groups by classifying its structural properties, such as disconnected, completeness, paths, cycles, and star. Abdussakir (2017) further developed the study of subgroup graphs by investigating the detour energy of the complement of the subgroup graph.

Additionally, Devi and Singh (2017) explore the cyclic subgroup graph of finite group and determine several of its domination numbers, such as the perfect domination number, strong triple connected domination number, two domination number, and more. Berlai and Ferov (2019) demonstrate that graph products preserve cyclic subgroup separability. Next, Rahin et al. (2020) introduced the non-normal cyclic subgroup graph, and the graph is then applied to

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some types of groups. For instance, Rahin (2024) determined the general form of the non-normal cyclic subgroup graph of dihedral and generalized quaternion groups. The non-normal cyclic subgroup graph of the quasidihedral group of order 16 has previously been investigated by Abdul Razak et al. (2026, in press). Therefore, this study extends the computation to the quasidihedral group of order 32 as a continuation of that work, with the aim of contributing to the establishment of the general form of the non-normal cyclic subgroup graph for quasidihedral groups.

In this study, two software packages are used to construct and visualize the non-normal cyclic subgroup graphs. The first software, which is Groups, Algorithms and Programming (GAP) software, is used to determine all non-normal cyclic subgroups of the quasidihedral group of order 32. GAP is a software system designed for computational discrete algebra, with a strong focus on Computational Group Theory. It is widely used in both research and education for exploring and analyzing mathematical structures such as groups, their representations, rings, vector spaces, algebras, combinatorial structures, codes, Lie algebras, and more (Rahin et al., 2022). Subsequently, the definition of the non-normal cyclic subgroup graph is applied to determine the directed edges between vertices, and the resulting graphs are visualized using Maple 2016.

The group presentation of quasidihedral group, QD_{2^n}, is given below (Humphreys, 1996):

$$QD_{2^n} = \langle a, b | a^{2^{n-1}} = b^2 = 1, bab = a^{2^{n-2}-1} \rangle$$

where $n = 5$.

Many researchers studied the quasidihedral group which applied to different scope of study. Najmuddin et al. (2019) examined the domination polynomial of commuting and noncommuting graphs associated with dihedral, generalized quaternion, and quasidihedral groups. Roslly et al. (2023) investigated the Randić index of the non-commuting graph associated with dihedral, generalized quaternion and quasidihedral groups by deriving their general formulas and presenting several computational examples. Then, Alimon et al. (2023) explored the Szegeb index of the non-commuting graphs associated with quasidihedral, dihedral and generalized quaternion groups. Furthermore, Alimon et al. (2025) studied on the Sombor index and Sombor polynomial of the power graphs for dihedral, generalized quaternion and quasidihedral groups.

This paper is structured into four sections. The introduction is presented in section 1, which consists of the introduction and literature review. Then, section 2 provides a preliminary, in which the relevant definitions of groups and graphs are presented. Section 3 contains the results and discussion, where the main findings on the non-normal cyclic subgroup graphs of the quasidihedral group of order 32 are presented. Finally, section 4 concludes the paper with a summary of the results.

Preliminaries

In this section, some related definitions of groups and graphs are given.

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Definition 1 Graph (Bondy & Murty, 1976)

A graph Γ is made up of (V, E), represented as $\Gamma = \big(V(\Gamma), E(\Gamma)\big)$. $V(\Gamma)$ is referred to as the vertices, while $E(\Gamma)$, a set of unordered pairs of vertices, is called the edges.

A graph Γ can be classified as either a directed graph or an undirected graph. A directed graph contains edges with specific directions, while an undirected graph contains edges without directions between vertices.

Definition 2 Subgroup Graph (Anderson et al., 2012)

Let G be a group and H is a subgroup of G. $\Gamma_H(G)$ denotes as the subgroup graph which is a directed graph with the vertex set G, and two distinct elements x and y are adjacent if and only if $xy \in H$.

Definition 3 Non-Normal Cyclic Subgroup Graph (Rahin et al., 2020)

Let $\Gamma_H^{NN}(G)$ be the non-normal cyclic subgroup graph of the non-normal cyclic subgroup of a group G, denotes as H. Then, $\Gamma_H^{NN}(G)$ is a directed graph with vertex set G such that x is the initial vertex and y is the terminal vertex of an edge if and only if $x \neq y$ and $xy \in H$, denoted by $x \to y$.

Example 1 Let $QD_{16} = \{1, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}$ with $a^8 = b^2 = 1, bab = a^3$ and the non-normal cyclic subgroups of QD_{16} are $\langle b\rangle, \langle a^2b\rangle, \langle a^4b\rangle, \langle a^6b\rangle, \langle b, a^4b\rangle, \langle a^2b, a^6b\rangle, \langle a^3b\rangle, \langle a^5b\rangle$. The non-normal cyclic subgroups were determined by using GAP (Groups, Algorithms, and Programming) software and the algorithm are presented in Appendix 1. First, let $H_1 = \langle b\rangle = \{1, b\}$. The product of the elements of QD_{16} and the directed edges are determined below and illustrated in Figure 1.

By Definition 3, $x \to y$ if and only if $x \neq y$ and $xy \in H$. Thus,

$a^5b \to a$ since $a^5b.a = b \in H_1$,
$a \to a^7b$ since $a.a^7b = b \in H_1$,
$a^3 \to a^5b$ since $a^3.a^5b = b \in H_1$,
$a^7b \to a^3$ since $a^7b.a^3 = b \in H_1$,
$ab \to a^5$ since $ab.a^5 = b \in H_1$,
$a^7 \to ab$ since $a^7.ab = b \in H_1$,
$a^5 \to a^3b$ since $a^5.a^3b = b \in H_1$,
$a^3b \to a^7$ since $a^3b.a^7 = b \in H_1$,
$a^6b \to a^6$ since $a^6b.a^6 = b \in H_1$,
$a^2 \to a^6b$ since $a^2.a^6b = b \in H_1$,
$a^6 \to a^2b$ since $a^6.a^2b = b \in H_1$,
$a^2b \to a^2$ since $a^2b.a^2 = b \in H_1$,

$1 \to b$ and $b \to 1$ since $1.b = b.1 = b \in H_1$,
$a^4 \to a^4b$ and $a^4b \to a^4$ since $a^4.a^4b = a^4b.a^4 = b \in H_1$,
$a \to a^7$ and $a^7 \to a$ since $a.a^7 = a^7.a = 1 \in H_1$,
$a^3 \to a^5$ and $a^5 \to a^3$ since $a^3.a^5 = a^5.a^3 = 1 \in H_1$,
$ab \to a^5b$ and $a^5b \to ab$ since $ab.a^5b = a^5b.ab = 1 \in H_1$,
$a^3b \to a^7b$ and $a^7b \to a^3b$ since $a^3b.a^7b = a^7b.a^3b = 1 \in H_1$,
$a^2 \to a^6$ and $a^6 \to a^2$ since $a^2.a^6 = a^6.a^2 = 1 \in H_1$.

Figure below shows the non-normal cyclic subgroup graph of QD_{16} which is constructed by using Maple software and the algorithm is presented in Appendix 2.

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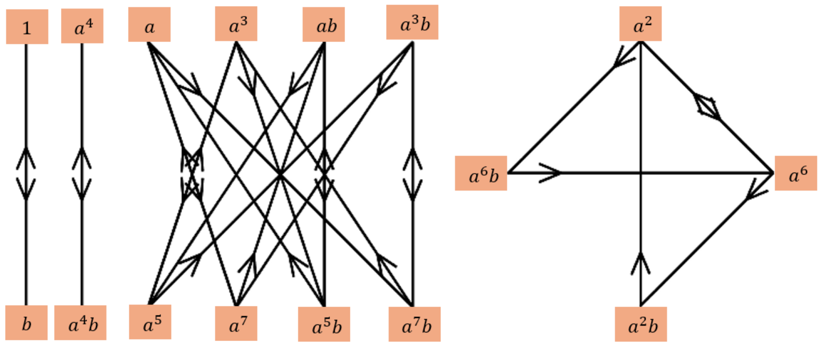

Figure 1: $\Gamma_{H_1}^{NN}(QD_{16})$

Results and Discussion

In this section, the non-normal cyclic subgroup graphs for the quasidihedral group of order 32 are determined. The methodology is presented in Figure 2.

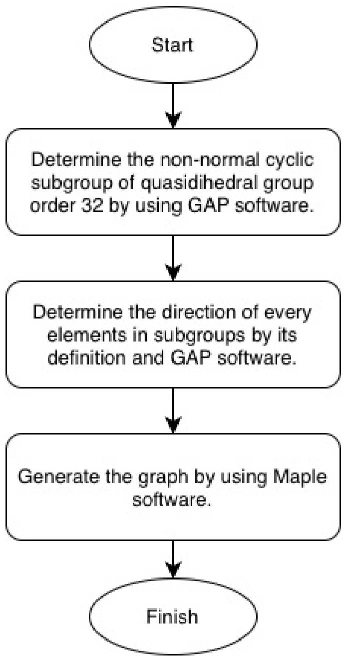

Figure 2: Research Methodology Flowchart

The quasidihedral groups of order 32, with its elements is $QD_{32} = \{1, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b, a^8b, a^9b, a^{10}b, a^{11}b, a^{12}b, a^{13}b, a^{14}b, a^{15}b\}$ with $a^{16} = b^2 = 1, bab = a^7$. The non-normal cyclic subgroups of QD_{32} are first determined by using GAP software. Then, the GAP code used to obtain the subgroups is presented in Figure 3, which shows that 20 non-normal cyclic subgroups of QD_{32} are not in simplified form.

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```
gap> F:=FreeGroup("a","b");
<free group on the generators [ a, b ]>
gap> a:=F.1;b:=F.2;
a
b
gap> G:=F/[a^16,b^2,b*a*b*a^9];
<fp group on the generators [ a, b ]>
gap> a:=G.1;b:=G.2;
a
b
gap> Order(G);
32
gap> e:=Elements(G);
[ <identity ...>, <[ [ 1, -2, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1 ], [ 2, 7 ] ]|a^-63>, b,
  <[ [ 1, -2, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1 ], [ 2, 6 ] ]|a^-54>,
  <[ [ 1, -2, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1 ], [ 2, 4 ] ]|a^-36>,
  <[ [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 8 ] ]|a^72>, a^-63*b,
  <[ [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 3 ] ]|a^27>,
  <[ [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 5 ] ]|a^45>,
  <[ [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2 ] ]|a^9>, a^54*b, a^36*b, a^72*b,
  <[ [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 6 ] ]|a^54>,
  <[ [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 2 ] ]|a^18>,
  <[ [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 4 ] ]|a^36>, a^-9*b, a^-27*b, a^9*b,
  <[ [ 1, -2, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1 ] ]|a^-9>,
  <[ [ 1, -2, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1 ], [ 2, 5 ] ]|a^-45>,
  <[ [ 1, -2, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1 ], [ 2, 3 ] ]|a^-27>, a^-54*b, a^-18*b, a^-36*b,
  <[ [ 1, -2, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1 ], [ 2, 2 ] ]|a^-18>, a^27*b, a^63*b, a^45*b,
  <[ [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 7 ] ]|a^63>, a^18*b, a^-45*b ]
gap> AllSubgroups(G);
[ Group([  ]), Group([ a^72 ]), Group([ b ]), Group([ a^-1*b*a ]), Group([ b^-1*a^-2*b*a^2*b ]),
  Group([ a^-4*b*a^4 ]), Group([ b^-1*a^-1*b*a*b ]), Group([ b^-1*a^-3*b*a^3*b ]), Group([ a^-2*b*a^2 ]),
  Group([ a^-3*b*a^3 ]), Group(<fp, no generators known>), Group([ b, a*b*a ]),
  Group([ a^-2*b^-1, a^-1*b*a, b*a^-1*b^-1*a^-1, a*b*a^-1*b^-1*a^-2 ]),
  Group([ b*a*b^-1*a, b*a^-1*b^-1*a^-1, a^3*b^-1*a^-1 ]), Group([ a^2*b^-1, a*b*a^-1, b*a*b^-1*a, a^-1*b*a*b^-1*a^2 ])
    , Group([ b*a^-1 ]), Group([ b*a, a^-2*b^-1*a^-1, a*b*a^-1*b^-1*a^-2 ]),
  Group([ a^-1*b*a^2, b*a^3, b*a^-1*b^-1*a^-1 ]), Group([ a*b*a^-2, b*a*b^-1*a, b*a^-3 ]),
  Group(<fp, no generators known>), Group([ b, a*b*a, a^-4 ]), Group([ a^2*b^-1, a*b*a^-1, a^-2*b^-1 ]),
  Group([ b*a^-1, a^-4 ]), Group([ b*a, a*b*a^-2, a^-4 ]), Group(<fp, no generators known>),
  Group(<fp, no generators known>), Group(<fp, no generators known>), Group([ a, b ]) ]
gap> N:=NormalSubgroups(G);
[ Group(<fp, no generators known>), Group(<fp, no generators known>),
  Group(<fp, no generators known>), Group(<fp, no generators known>),
  Group(<fp, no generators known>), Group(<fp, no generators known>),
  Group(<fp, no generators known>), Group(<fp, no generators known>) ]
gap> Elements(N);
[ Group([ a^-1*b*a^-2*b^-1*a^-1, b*a^2*b^-1*a^2, b*a^-2*b^-1*a^-2, a*b*a^-2*b^-1*a^-3 ]),
  Group([ a, b ]), Group([ a ]), Group([ a^-2, b ]), Group([ a^-2, b*a^-1 ]), Group([ a^-2 ]),
  Group([ a^-4 ]), Group([ b*a*b^-1*a ]) ]
```

Figure 3: GAP Coding to Obtain the Non-Normal Cyclic Subgroups of QD_{32}

For each non-normal cyclic subgroup, a non-normal cyclic subgroup graph can be constructed. After generating all such graphs, a comparative analysis reveals that the resulting graphs can be classified into four different types. In this paper, only these four types of non-normal cyclic subgroup graphs are presented and discussed, with one representative subgroup chosen for each type. Table 1 presents those four representative non-normal cyclic subgroups.

Table 1: The Elements of the Non-Normal Cyclic Subgroups of QD_{32}

Subgroup generated by GAP software	Simplified subgroup	Elements
$\langle b\rangle$	$\langle b\rangle$	$\{1, b\}$
$\langle b, aba\rangle$	$\langle b, a^8b\rangle$	$\{1, b, a^8, a^8b\}$
$\langle b, a^{-4}, aba\rangle$	$\langle b, a^{12}, a^8b\rangle$	$\{1, a^4, a^8, a^{12}, b, a^4b, a^8b, a^{12}b\}$
$\langle a^{-2}b^{-1}, a^{-1}ba, ba^{-1}b^{-1}a^{-1}, aba^{-1}b^{-1}a^{-2}\rangle$	$\langle a^8, a^{12}, a^6b, a^{14}b\rangle$	$\{1, a^4, a^6, a^8, a^{12}, a^{14}, b, a^4b, a^5b, a^6b, a^8b, a^{12}b, a^{14}b\}$

After all the non-normal cyclic subgroups of QD_{32} are identified, the determination on the direction of elements in subgroups is shown in the following propositions.

Proposition 1, Let $\Gamma_{H_1}^{NN}(QD_{32})$ be the non-normal cyclic subgroup graph of QD_{32}, where $H_1 = \langle b\rangle = \{1, b\}$. Then, $\Gamma_{H_1}^{NN}(QD_{32})$ is as shown in Figure 4.

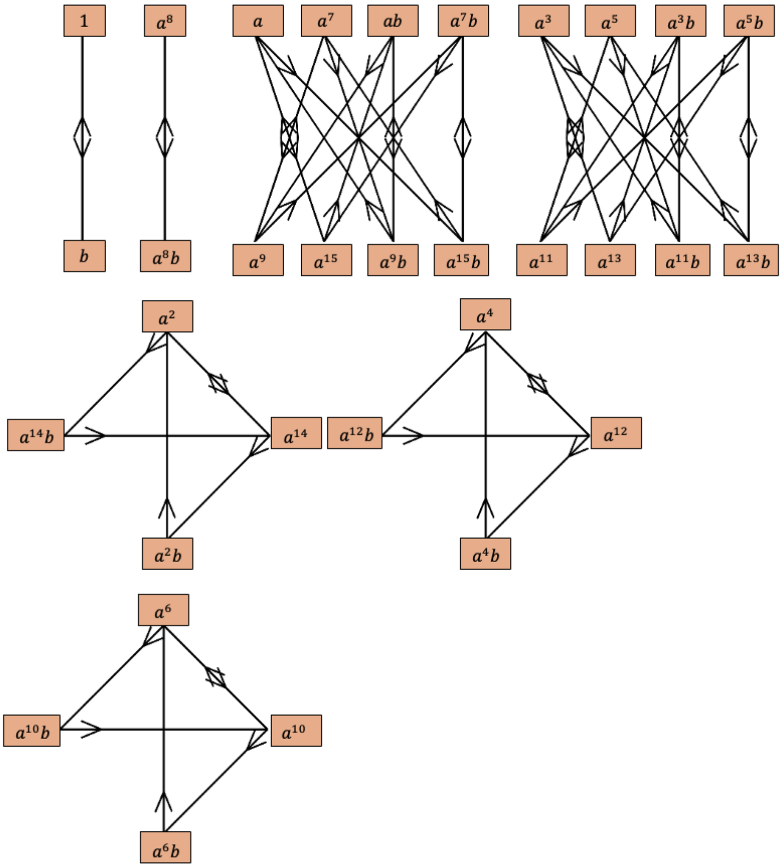

Figure 4: $\Gamma_{H_1}^{NN}(QD_{32})$

Proof. Let $H_1 = \langle b \rangle = \{1, b\}$. The directions of every elements in subgroups are determined by definition and GAP software as shown in Figure 5.


```
gap> H1:=Subgroup(G,[b]);
Group([ b ])
gap> Elements(H1);
[ <identity ...>, <[ [ 1, 1 ] ]|b> ]
gap> for i in e do
> for j in e do;
> if i<>j then
> c:=i*j;
> d:=c in H1;
> if (d=true) then
> Print ("{",i,",",j,",",c,",",d,"},");
> fi;
> fi;
> od;
> od;
{<identity ...>,b,b,true},{a^-63,a^63*b,b,true},{a^-63,a^63,<identity ...>,true},{b,<identity ...>,b,\
true},{a^-54,a^54*b,b,true},{a^-54,a^54,<identity ...>,true},{a^-36,a^36*b,b,true},{a^-36,a^36,<ident\
ity ...>,true},{a^72,a^72*b,a^144*b,true},{a^-63*b,a^9,a^-63*b*a^9,true},{a^-63*b,a^9*b,a^-63*b*a^9*b\
,true},{a^27,a^-27*b,b,true},{a^27,a^-27,<identity ...>,true},{a^45,a^-45,<identity ...>,true},{a^45,\
a^-45*b,b,true},{a^9,a^-9*b,b,true},{a^9,a^-9,<identity ...>,true},{a^54*b,a^54,a^54*b*a^54,true},{a^\
36*b,a^36,a^36*b*a^36,true},{a^72*b,a^72,a^72*b*a^72,true},{a^54,a^-54,<identity ...>,true},{a^54,a^-\
54*b,b,true},{a^18,a^-18*b,b,true},{a^18,a^-18,<identity ...>,true},{a^36,a^-36,<identity ...>,true},\
{a^36,a^-36*b,b,true},{a^-9*b,a^63*b,a^-9*b*a^63*b,true},{a^-9*b,a^63,a^-9*b*a^63,true},{a^-27*b,a^45\
,a^-27*b*a^45,true},{a^-27*b,a^45*b,a^-27*b*a^45*b,true},{a^9*b,a^-63,a^9*b*a^-63,true},{a^9*b,a^-63*\
b,a^9*b*a^-63*b,true},{a^-9,a^9,<identity ...>,true},{a^-9,a^9*b,b,true},{a^-45,a^45,<identity ...>,t\
rue},{a^-45,a^45*b,b,true},{a^-27,a^27,<identity ...>,true},{a^-27,a^27*b,b,true},{a^-54*b,a^-54,a^-5\
4*b*a^-54,true},{a^-18*b,a^-18,a^-18*b*a^-18,true},{a^-36*b,a^-36,a^-36*b*a^-36,true},{a^-18,a^18,<id\
entity ...>,true},{a^-18,a^18*b,b,true},{a^27*b,a^-45,a^27*b*a^-45,true},{a^27*b,a^-45*b,a^27*b*a^-45\
*b,true},{a^63*b,a^-9*b,a^63*b*a^-9*b,true},{a^63*b,a^-9,a^63*b*a^-9,true},{a^45*b,a^-27*b,a^45*b*a^-\
27*b,true},{a^45*b,a^-27,a^45*b*a^-27,true},{a^63,a^-63,<identity ...>,true},{a^63,a^-63*b,b,true},{a\
^18*b,a^18,a^18*b*a^18,true},{a^-45*b,a^27,a^-45*b*a^27,true},{a^-45*b,a^27*b,a^-45*b*a^27*b,true},
```
Figure 5: GAP Coding to Determine the Direction of Elements in Subgroup $\langle b \rangle$ of QD_{32}

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By Definition 3,

$a^{15} \to a$ since $a^{15} \cdot a = 1 \in H_1$
$a \to a^{15}$ since $a \cdot a^{15} = 1 \in H_1$
$a^{14} \to a^2$ since $a^{14} \cdot a^2 = 1 \in H_1$
$a^2 \to a^{14}$ since $a^2 \cdot a^{14} = 1 \in H_1$
$a^{13} \to a^3$ since $a^{13} \cdot a^3 = 1 \in H_1$
$a^3 \to a^{13}$ since $a^3 \cdot a^{13} = 1 \in H_1$
$a^{12} \to a^4$ since $a^{12} \cdot a^4 = 1 \in H_1$
$a^4 \to a^{12}$ since $a^4 \cdot a^{12} = 1 \in H_1$
$a^{11} \to a^5$ since $a^{11} \cdot a^5 = 1 \in H_1$
$a^5 \to a^{11}$ since $a^5 \cdot a^{11} = 1 \in H_1$
$a^{10} \to a^6$ since $a^{10} \cdot a^6 = 1 \in H_1$
$a^6 \to a^{10}$ since $a^6 \cdot a^{10} = 1 \in H_1$
$a^9 \to a^7$ since $a^9 \cdot a^7 = 1 \in H_1$
$a^7 \to a^9$ since $a^7 \cdot a^9 = 1 \in H_1$
$a^9b \to ab$ since $a^9b \cdot ab = 1 \in H_1$
$ab \to a^9b$ since $ab \cdot a^9b = 1 \in H_1$
$a^{11}b \to a^3b$ since $a^{11}b \cdot a^3b = 1 \in H_1$
$a^3b \to a^{11}b$ since $a^3b \cdot a^{11}b = 1 \in H_1$
$a^{13}b \to a^5b$ since $a^{13}b \cdot a^5b = 1 \in H_1$
$a^5b \to a^{13}b$ since $a^5b \cdot a^{13}b = 1 \in H_1$
$a^{15}b \to a^7b$ since $a^{15}b \cdot a^7b = 1 \in H_1$
$a^7b \to a^{15}b$ since $a^7b \cdot a^{15}b = 1 \in H_1$
$b \to 1$ since $b \cdot 1 = b \in H_1$
$1 \to b$ since $1 \cdot b = b \in H_1$
$a^8b \to a^8$ since $a^8b \cdot a^8 = b \in H_1$
$a^8 \to a^8b$ since $a^8 \cdot a^8b = b \in H_1$
$a^9b \to a$ since $a^9b \cdot a = b \in H_1$

$a^2b \to a^2$ since $a^2b \cdot a^2 = b \in H_1$
$a^{11}b \to a^3$ since $a^{11}b \cdot a^3 = b \in H_1$
$a^4b \to a^4$ since $a^4b \cdot a^4 = b \in H_1$
$a^{13}b \to a^5$ since $a^{13}b \cdot a^5 = b \in H_1$
$a^6b \to a^6$ since $a^6b \cdot a^6 = b \in H_1$
$a^{15}b \to a^7$ since $a^{15}b \cdot a^7 = b \in H_1$
$ab \to a^9$ since $ab \cdot a^9 = b \in H_1$
$a^{10}b \to a^{10}$ since $a^{10}b \cdot a^{10} = b \in H_1$
$a^3b \to a^{11}$ since $a^3b \cdot a^{11} = b \in H_1$
$a^{12}b \to a^{12}$ since $a^{12}b \cdot a^{12} = b \in H_1$
$a^5b \to a^{13}$ since $a^5b \cdot a^{13} = b \in H_1$
$a^{14}b \to a^{14}$ since $a^{14}b \cdot a^{14} = b \in H_1$
$a^7b \to a^{15}$ since $a^7b \cdot a^{15} = b \in H_1$
$a^{15} \to ab$ since $a^{15} \cdot ab = b \in H_1$
$a^{14} \to a^2b$ since $a^{14} \cdot a^2b = b \in H_1$
$a^{13} \to a^3b$ since $a^{13} \cdot a^3b = b \in H_1$
$a^{12} \to a^4b$ since $a^{12} \cdot a^4b = b \in H_1$
$a^{11} \to a^5b$ since $a^{11} \cdot a^5b = b \in H_1$
$a^{10} \to a^6b$ since $a^{10} \cdot a^6b = b \in H_1$
$a^9 \to a^7b$ since $a^9 \cdot a^7b = b \in H_1$
$a^7 \to a^9b$ since $a^7 \cdot a^9b = b \in H_1$
$a^6 \to a^{10}b$ since $a^6 \cdot a^{10}b = b \in H_1$
$a^5 \to a^{11}b$ since $a^5 \cdot a^{11}b = b \in H_1$
$a^4 \to a^{12}b$ since $a^4 \cdot a^{12}b = b \in H_1$
$a^3 \to a^{13}b$ since $a^3 \cdot a^{13}b = b \in H_1$
$a^2 \to a^{14}b$ since $a^2 \cdot a^{14}b = b \in H_1$
$a \to a^{15}b$ since $a \cdot a^{15}b = b \in H_1$

Therefore, $\Gamma_{H_1}^{NN}(QD_{32})$ is the graph shown in Figure 4.

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Proposition 2, Let $\Gamma_{H_2}^{NN}(QD_{32})$ be the non-normal cyclic subgroup graph of QD_{32}, where $H_2 = \langle b, a^8 b\rangle = \{1, b, a^8, a^8 b\}$. Then, $\Gamma_{H_2}^{NN}(QD_{32})$ is as shown in Figure 6.

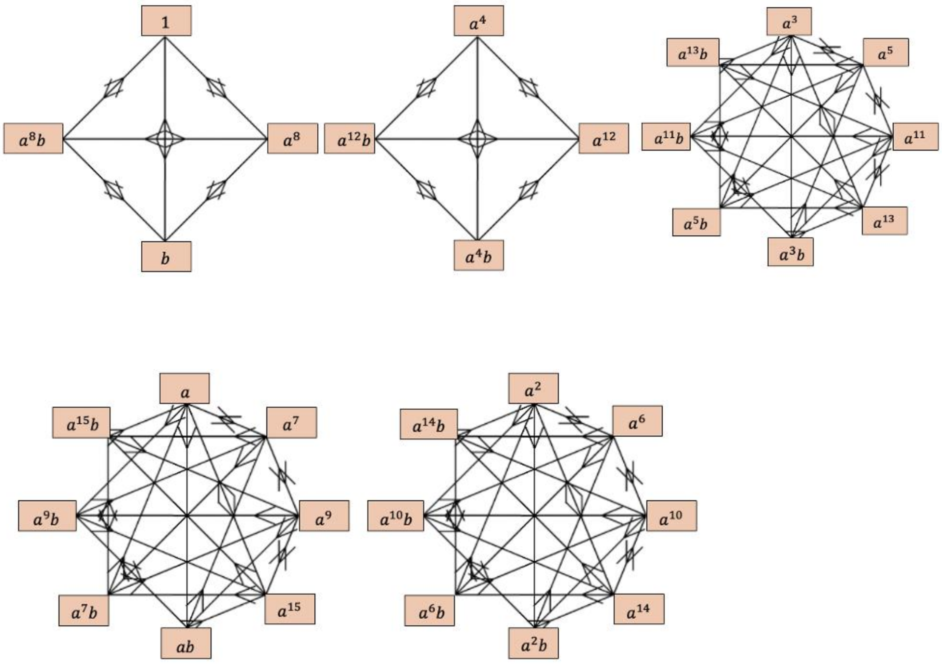

Figure 6: $\Gamma_{H_2}^{NN}(QD_{32})$

Proof. Let $H_2 = \langle b, a^8 b\rangle = \{1, b, a^8, a^8 b\}$. The directions of every elements in subgroups are determined by definition and GAP software as shown in Figure 6.

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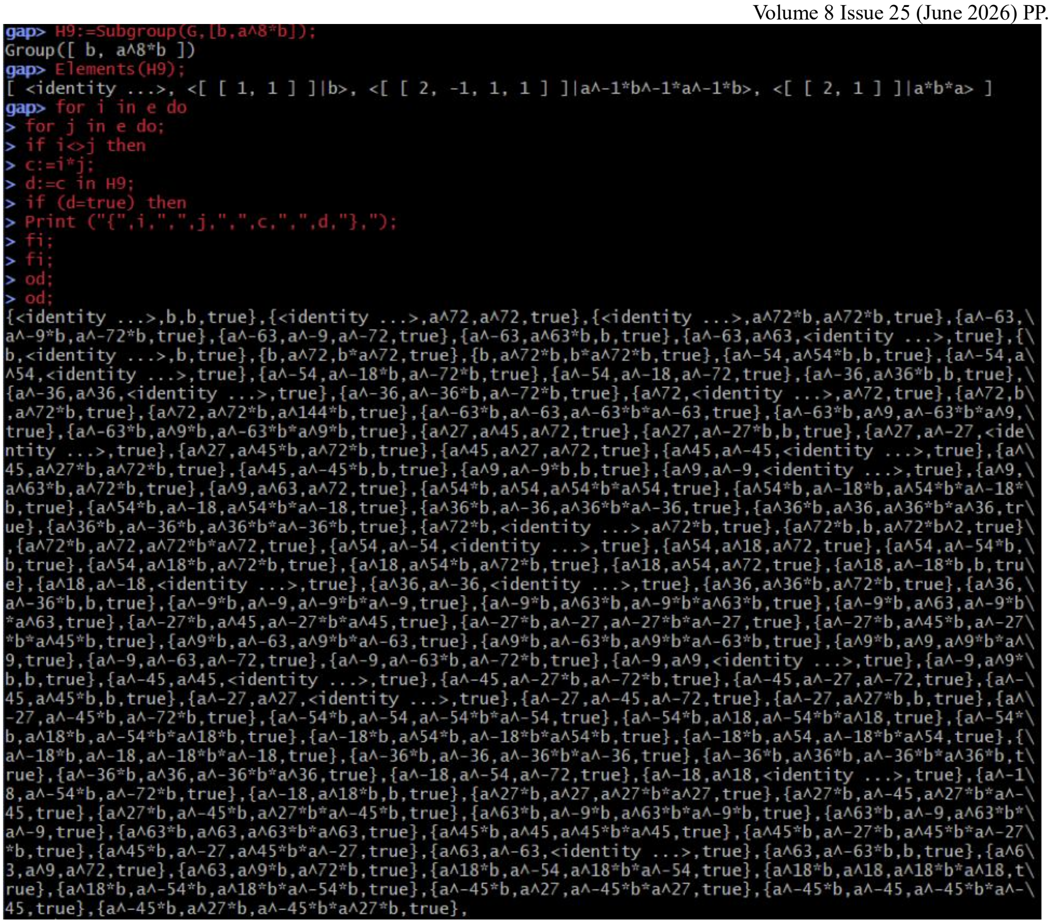

```
gap> H9:=Subgroup(G,[b,a^8*b]);
Group([ b, a^8*b ])
gap> Elements(H9);
[ <identity ...>, <[ [ 1, 1 ] ]|b>, <[ [ 2, -1, 1, 1 ] ]|a^-1*b^-1*a^-1*b>, <[ [ 2, 1 ] ]|a*b*a> ]
gap> for i in e do
> for j in e do;
> if i<>j then
> c:=i*j;
> d:=c in H9;
> if (d=true) then
> Print ("{",i,",",j,",",c,",",d,"},");
> fi;
> fi;
> od;
> od;
{<identity ...>,b,b,true},{<identity ...>,a^72,a^72,true},{<identity ...>,a^72*b,a^72*b,true},{a^-63,\
a^-9*b,a^-72*b,true},{a^-63,a^-9,a^-72,true},{a^-63,a^63*b,b,true},{a^-63,a^63,<identity ...>,true},{\
b,<identity ...>,b,true},{b,a^72,b*a^72,true},{b,a^72*b,b*a^72*b,true},{a^-54,a^54*b,b,true},{a^-54,a\
^54,<identity ...>,true},{a^-54,a^-18*b,a^-72*b,true},{a^-54,a^-18,a^-72,true},{a^-36,a^36*b,b,true},\
{a^-36,a^36,<identity ...>,true},{a^-36,a^-36*b,a^-72*b,true},{a^72,<identity ...>,a^72,true},{a^72,b\
,a^72*b,true},{a^72,a^72*b,a^144*b,true},{a^-63*b,a^-63,a^-63*b*a^-63,true},{a^-63*b,a^9,a^-63*b*a^9,\
true},{a^-63*b,a^9*b,a^-63*b*a^9*b,true},{a^27,a^45,a^72,true},{a^27,a^-27*b,b,true},{a^27,a^-27,<ide\
ntity ...>,true},{a^27,a^45*b,a^72*b,true},{a^45,a^27,a^72,true},{a^45,a^-45,<identity ...>,true},{a^\
45,a^27*b,a^72*b,true},{a^45,a^-45*b,b,true},{a^9,a^-9*b,b,true},{a^9,a^-9,<identity ...>,true},{a^9,\
a^63*b,a^72*b,true},{a^9,a^63,a^72,true},{a^54*b,a^54,a^54*b*a^54,true},{a^54*b,a^-18*b,a^54*b*a^-18*\
b,true},{a^54*b,a^-18,a^54*b*a^-18,true},{a^36*b,a^-36,a^36*b*a^-36,true},{a^36*b,a^36,a^36*b*a^36,tr\
ue},{a^36*b,a^-36*b,a^36*b*a^-36*b,true},{a^72*b,<identity ...>,a^72*b,true},{a^72*b,b,a^72*b^2,true}\
,{a^72*b,a^72,a^72*b*a^72,true},{a^54,a^-54,<identity ...>,true},{a^54,a^18,a^72,true},{a^54,a^-54*b,\
b,true},{a^54,a^18*b,a^72*b,true},{a^18,a^54*b,a^72*b,true},{a^18,a^54,a^72,true},{a^18,a^-18*b,b,tru\
e},{a^18,a^-18,<identity ...>,true},{a^36,a^-36,<identity ...>,true},{a^36,a^36*b,a^72*b,true},{a^36,\
a^-36*b,b,true},{a^-9*b,a^-9,a^-9*b*a^-9,true},{a^-9*b,a^63*b,a^-9*b*a^63*b,true},{a^-9*b,a^63,a^-9*b\
*a^63,true},{a^-27*b,a^45,a^-27*b*a^45,true},{a^-27*b,a^-27,a^-27*b*a^-27,true},{a^-27*b,a^45*b,a^-27\
*b*a^45*b,true},{a^9*b,a^-63,a^9*b*a^-63,true},{a^9*b,a^-63*b,a^9*b*a^-63*b,true},{a^9*b,a^9,a^9*b*a^\
9,true},{a^-9,a^-63,a^-72,true},{a^-9,a^-63*b,a^-72*b,true},{a^-9,a^9,<identity ...>,true},{a^-9,a^9*\
b,b,true},{a^-45,a^45,<identity ...>,true},{a^-45,a^-27*b,a^-72*b,true},{a^-45,a^-27,a^-72,true},{a^-\
45,a^45*b,b,true},{a^-27,a^27,<identity ...>,true},{a^-27,a^-45,a^-72,true},{a^-27,a^27*b,b,true},{a^\
-27,a^-45*b,a^-72*b,true},{a^-54*b,a^-54,a^-54*b*a^-54,true},{a^-54*b,a^18,a^-54*b*a^18,true},{a^-54*\
b,a^18*b,a^-54*b*a^18*b,true},{a^-18*b,a^54*b,a^-18*b*a^54*b,true},{a^-18*b,a^54,a^-18*b*a^54,true},{\
a^-18*b,a^-18,a^-18*b*a^-18,true},{a^-36*b,a^-36,a^-36*b*a^-36,true},{a^-36*b,a^36*b,a^-36*b*a^36*b,t\
rue},{a^-36*b,a^36,a^-36*b*a^36,true},{a^-18,a^-54,a^-72,true},{a^-18,a^18,<identity ...>,true},{a^-1\
8,a^-54*b,a^-72*b,true},{a^-18,a^18*b,b,true},{a^27*b,a^27,a^27*b*a^27,true},{a^27*b,a^-45,a^27*b*a^-\
45,true},{a^27*b,a^-45*b,a^27*b*a^-45*b,true},{a^63*b,a^-9*b,a^63*b*a^-9*b,true},{a^63*b,a^-9,a^63*b*\
a^-9,true},{a^63*b,a^63,a^63*b*a^63,true},{a^45*b,a^45,a^45*b*a^45,true},{a^45*b,a^-27*b,a^45*b*a^-27\
*b,true},{a^45*b,a^-27,a^45*b*a^-27,true},{a^63,a^-63,<identity ...>,true},{a^63,a^-63*b,b,true},{a^6\
3,a^9,a^72,true},{a^63,a^9*b,a^72*b,true},{a^18*b,a^-54,a^18*b*a^-54,true},{a^18*b,a^18,a^18*b*a^18,t\
rue},{a^18*b,a^-54*b,a^18*b*a^-54*b,true},{a^-45*b,a^27,a^-45*b*a^27,true},{a^-45*b,a^-45,a^-45*b*a^-\
45,true},{a^-45*b,a^27*b,a^-45*b*a^27*b,true},
```

Figure 5: GAP Coding to Determine the Direction of Elements in Subgroup $\langle b, a^8 b\rangle$ of QD_{32}

By Definition 3,

$a^{15} \rightarrow a$ since $a^{15} \cdot a = 1 \in H_2$

$a \rightarrow a^{15}$ since $a \cdot a^{15} = 1 \in H_2$

$a^{14} \rightarrow a^2$ since $a^{14} \cdot a^2 = 1 \in H_2$

$a^2 \rightarrow a^{14}$ since $a^2 \cdot a^{14} = 1 \in H_2$

$a^{13} \rightarrow a^3$ since $a^{13} \cdot a^3 = 1 \in H_2$

$a^3 \rightarrow a^{13}$ since $a^3 \cdot a^{13} = 1 \in H_2$

$a^{12} \rightarrow a^4$ since $a^{12} \cdot a^4 = 1 \in H_2$

$a^4 \rightarrow a^{12}$ since $a^4 \cdot a^{12} = 1 \in H_2$

$a^{11} \rightarrow a^5$ since $a^{11} \cdot a^5 = 1 \in H_2$

$a^5 \rightarrow a^{11}$ since $a^5 \cdot a^{11} = 1 \in H_2$

$a^{10} \rightarrow a^6$ since $a^{10} \cdot a^6 = 1 \in H_2$

$a^6 \rightarrow a^{10}$ since $a^6 \cdot a^{10} = 1 \in H_2$

$a \rightarrow a^{15}b$ since $a \cdot a^{15}b = b \in H_2$

$a^8 \rightarrow 1$ since $a^8 \cdot 1 = a^8 \in H_2$

$1 \rightarrow a^8$ since $1 \cdot a^8 = a^8 \in H_2$

$a^7 \rightarrow a$ since $a^7 \cdot a = a^8 \in H_2$

$a \rightarrow a^7$ since $a \cdot a^7 = a^8 \in H_2$

$a^6 \rightarrow a^2$ since $a^6 \cdot a^2 = a^8 \in H_2$

$a^2 \rightarrow a^6$ since $a^2 \cdot a^6 = a^8 \in H_2$

$a^5 \rightarrow a^3$ since $a^5 \cdot a^3 = a^8 \in H_2$

$a^3 \rightarrow a^5$ since $a^3 \cdot a^5 = a^8 \in H_2$

$a^{15} \rightarrow a^9$ since $a^{15} \cdot a^9 = a^8 \in H_2$

$a^9 \rightarrow a^{15}$ since $a^9 \cdot a^{15} = a^8 \in H_2$

$a^{14} \rightarrow a^{10}$ since $a^{14} \cdot a^{10} = a^8 \in H_2$

$a^9 \rightarrow a^7$ since $a^9 \cdot a^7 = 1 \in H_2$

$a^7 \rightarrow a^9$ since $a^7 \cdot a^9 = 1 \in H_2$

$a^9b \rightarrow ab$ since $a^9b \cdot ab = 1 \in H_2$

$ab \rightarrow a^9b$ since $ab \cdot a^9b = 1 \in H_2$

$a^{11}b \rightarrow a^3b$ since $a^{11}b \cdot a^3b = 1 \in H_2$

$a^3b \rightarrow a^{11}b$ since $a^3b \cdot a^{11}b = 1 \in H_2$

$a^{13}b \rightarrow a^5b$ since $a^{13}b \cdot a^5b = 1 \in H_2$

$a^5b \rightarrow a^{13}b$ since $a^5b \cdot a^{13}b = 1 \in H_2$

$a^{15}b \rightarrow a^7b$ since $a^{15}b \cdot a^7b = 1 \in H_2$

$a^7b \rightarrow a^{15}b$ since $a^7b \cdot a^{15}b = 1 \in H_2$

$b \rightarrow 1$ since $b \cdot 1 = b \in H_2$

$1 \rightarrow b$ since $1 \cdot b = b \in H_2$

$a^8b \rightarrow a^8$ since $a^8b \cdot a^8 = b \in H_2$

$a^8 \rightarrow a^8b$ since $a^8 \cdot a^8b = b \in H_2$

$a^9b \rightarrow a$ since $a^9b \cdot a = b \in H_2$

$a^2b \rightarrow a^2$ since $a^2b \cdot a^2 = b \in H_2$

$a^{11}b \rightarrow a^3$ since $a^{11}b \cdot a^3 = b \in H_2$

$a^4b \rightarrow a^4$ since $a^4b \cdot a^4 = b \in H_2$

$a^{13}b \rightarrow a^5$ since $a^{13}b \cdot a^5 = b \in H_2$

$a^6b \rightarrow a^6$ since $a^6b \cdot a^6 = b \in H_2$

$a^{15}b \rightarrow a^7$ since $a^{15}b \cdot a^7 = b \in H_2$

$ab \rightarrow a^9$ since $ab \cdot a^9 = b \in H_2$

$a^{10}b \rightarrow a^{10}$ since $a^{10}b \cdot a^{10} = b \in H_2$

$a^3b \rightarrow a^{11}$ since $a^3b \cdot a^{11} = 1 \in H_2$

$a^{12}b \rightarrow a^{12}$ since $a^{12}b \cdot a^{12} = b \in H_2$

$a^5b \rightarrow a^{13}$ since $a^5b \cdot a^{13} = b \in H_2$

$a^{14}b \rightarrow a^{14}$ since $a^{14}b \cdot a^{14} = b \in H_2$

$a^7b \rightarrow a^{15}$ since $a^7b \cdot a^{15} = b \in H_2$

$a^{15} \rightarrow ab$ since $a^{15} \cdot ab = b \in H_2$

$a^{14} \rightarrow a^2b$ since $a^{14} \cdot a^2b = b \in H_2$

$a^{13} \rightarrow a^3b$ since $a^{13} \cdot a^3b = b \in H_2$

$a^{10} \rightarrow a^{14}$ since $a^{10} \cdot a^{14} = a^8 \in H_2$

$a^{13} \rightarrow a^{11}$ since $a^{13} \cdot a^{11} = a^8 \in H_2$

$a^{11} \rightarrow a^{13}$ since $a^{11} \cdot a^{13} = a^8 \in H_2$

$a^8b \rightarrow b$ since $a^8b \cdot b = a^8 \in H_2$

$b \rightarrow a^8b$ since $b \cdot a^8b = a^8 \in H_2$

$a^{10}b \rightarrow a^2b$ since $a^{10}b \cdot a^2b = a^8 \in H_2$

$a^2b \rightarrow a^{10}b$ since $a^2b \cdot a^{10}b = a^8 \in H_2$

$a^{12}b \rightarrow a^4b$ since $a^{12}b \cdot a^4b = a^8 \in H_2$

$a^4b \rightarrow a^{12}b$ since $a^4b \cdot a^{12}b = a^8 \in H_2$

$a^{14}b \rightarrow a^6b$ since $a^{14}b \cdot a^6b = a^8 \in H_2$

$a^6b \rightarrow a^{14}b$ since $a^6b \cdot a^{14}b = a^8 \in H_2$

$a^8b \rightarrow 1$ since $a^8b \cdot 1 = a^8b \in H_2$

$1 \rightarrow a^8b$ since $1 \cdot a^8b = a^8b \in H_2$

$b \rightarrow a^8$ since $b \cdot a^8 = a^8b \in H_2$

$a^8 \rightarrow b$ since $a^8 \cdot b = a^8b \in H_2$

$ab \rightarrow a$ since $ab \cdot a = a^8b \in H_2$

$a^{10}b \rightarrow a^2$ since $a^{10}b \cdot a^2 = a^8b \in H_2$

$a^3b \rightarrow a^3$ since $a^3b \cdot a^3 = a^8b \in H_2$

$a^{12}b \rightarrow a^4$ since $a^{12}b \cdot a^4 = a^8b \in H_2$

$a^5b \rightarrow a^5$ since $a^5b \cdot a^5 = a^8b \in H_2$

$a^{14}b \rightarrow a^6$ since $a^{14}b \cdot a^6 = a^8b \in H_2$

$a^7b \rightarrow a^7$ since $a^7b \cdot a^7 = a^8b \in H_2$

$a^9b \rightarrow a^9$ since $a^9b \cdot a^9 = a^8b \in H_2$

$a^2b \rightarrow a^{10}$ since $a^2b \cdot a^{10} = a^8b \in H_2$

$a^{11}b \rightarrow a^{11}$ since $a^{11}b \cdot a^{11} = a^8b \in H_2$

$a^4b \rightarrow a^{12}$ since $a^4b \cdot a^{12} = a^8b \in H_2$

$a^{13}b \rightarrow a^{13}$ since $a^{13}b \cdot a^{13} = a^8b \in H_2$

$a^6b \rightarrow a^{14}$ since $a^6b \cdot a^{14} = a^8b \in H_2$

$a^{15}b \rightarrow a^{15}$ since $a^{15}b \cdot a^{15} = a^8b \in H_2$

$a^7 \rightarrow ab$ since $a^7 \cdot ab = a^8b \in H_2$

$a^6 \rightarrow a^2b$ since $a^6 \cdot a^2b = a^8b \in H_2$

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$a^{12} \to a^{4}b$ since $a^{12} \cdot a^{4}b = b \in H_2$

$a^{11} \to a^{5}b$ since $a^{11} \cdot a^{5}b = b \in H_2$

$a^{10} \to a^{6}b$ since $a^{10} \cdot a^{6}b = b \in H_2$

$a^{9} \to a^{7}b$ since $a^{9} \cdot a^{7}b = b \in H_2$

$a^{7} \to a^{9}b$ since $a^{7} \cdot a^{9}b = b \in H_2$

$a^{6} \to a^{10}b$ since $a^{6} \cdot a^{10}b = b \in H_2$

$a^{5} \to a^{11}b$ since $a^{5} \cdot a^{11}b = b \in H_2$

$a^{4} \to a^{12}b$ since $a^{4} \cdot a^{12}b = b \in H_2$

$a^{3} \to a^{13}b$ since $a^{3} \cdot a^{13}b = b \in H_2$

$a^{2} \to a^{14}b$ since $a^{2} \cdot a^{14}b = b \in H_2$

$a^{10} \to a^{14}b$ since $a^{10} \cdot a^{14}b = a^{8}b \in H_2$

$a^{5} \to a^{3}b$ since $a^{5} \cdot a^{3}b = a^{8}b \in H_2$

$a^{4} \to a^{4}b$ since $a^{4} \cdot a^{4}b = a^{8}b \in H_2$

$a^{3} \to a^{5}b$ since $a^{3} \cdot a^{5}b = a^{8}b \in H_2$

$a^{2} \to a^{6}b$ since $a^{2} \cdot a^{6}b = a^{8}b \in H_2$

$a \to a^{7}b$ since $a \cdot a^{7}b = a^{8}b \in H_2$

$a^{15} \to a^{9}b$ since $a^{15} \cdot a^{9}b = a^{8}b \in H_2$

$a^{14} \to a^{10}b$ since $a^{14} \cdot a^{10}b = a^{8}b \in H_2$

$a^{13} \to a^{11}b$ since $a^{13} \cdot a^{11}b = a^{8}b \in H_2$

$a^{12} \to a^{12}b$ since $a^{12} \cdot a^{12}b = a^{8}b \in H_2$

$a^{11} \to a^{13}b$ since $a^{11} \cdot a^{13}b = a^{8}b \in H_2$

$a^{9} \to a^{15}b$ since $a^{9} \cdot a^{15}b = a^{8}b \in H_2$

Therefore, $\Gamma_{H_2}^{NN}(QD_{32})$ is the graph shown in Figure 6.

Proposition 3, Let $\Gamma_{H_3}^{NN}(QD_{32})$ be the non-normal cyclic subgroup graph of QD_{32}, where $H_3 = \langle b, a^{12}, a^{8}b\rangle = \{1, a^{4}, a^{8}, a^{12}, b, a^{4}b, a^{8}b, a^{12}b\}$. Then, $\Gamma_{H_3}^{NN}(QD_{32})$ is as shown in Figure 8.

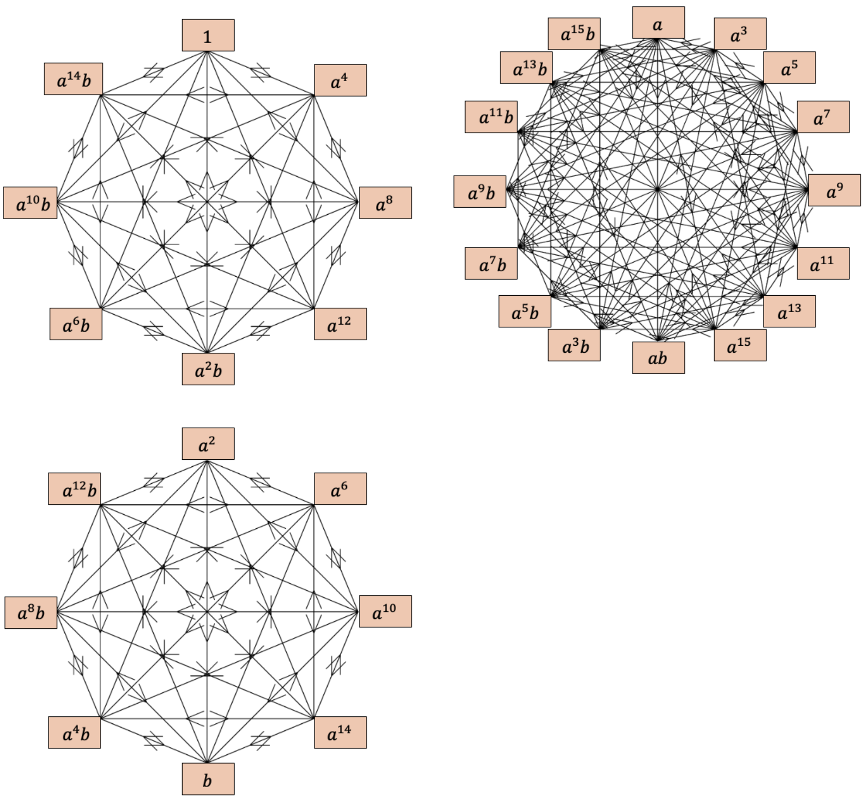

Figure 8: $\Gamma_{H_3}^{NN}(QD_{32})$

Proof. Let $H_3 = \langle b, a^{8}b, a^{12}\rangle = \{1, a^{4}, a^{8}, a^{12}, b, a^{4}b, a^{8}b, a^{12}b\}$. The directions of every elements in subgroups are determined by definition and GAP software as shown in Figure 9.

```
gap> H17:=Subgroup(G,[b,a^8*b,a^12]);
Group([ b, a^8*b, a^12 ])
gap> Elements(H17);
[ <identity ...>, <[ [ 3, -1, 3, -1, 2, -1 ] ]|a^7*b^-1*a^-1>, <[ [ 3, 1 ] ]|a^-4>,
  <[ [ 3, 2 ] ]|a^-8>, <[ [ 3, -2, 3, -1, 2, -1 ] ]|a^11*b^-1*a^-1>,
  <[ [ 3, -3, 3, -1, 2, -1 ] ]|a^15*b^-1*a^-1>, <[ [ 3, -1 ] ]|a^4>,
  <[ [ 3, -1, 2, -1 ] ]|a^3*b^-1*a^-1> ]
gap> for i in e do
> for j in e do;
> if i<>j then
> c:=i*j;
> d:=c in H17;
> if (d=true) then
> Print ("{",i,",",j,",",c,",",d,"},");
> fi;
>
> fi;
> od;
> od;
{<identity ...>,b,b,true},{<identity ...>,a^-36,a^-36,true},{<identity ...>,a^72,a^72,true},{<identit\
y ...>,a^36*b,a^36*b,true},{<identity ...>,a^72*b,a^72*b,true},{<identity ...>,a^36,a^36,true},{<iden\
tity ...>,a^-36*b,a^-36*b,true},{a^-63,a^27,a^-36,true},{a^-63,a^-9*b,a^-72*b,true},{a^-63,a^-9,a^-72\
,true},{a^-63,a^-45,a^-108,true},{a^-63,a^27*b,a^-36*b,true},{a^-63,a^63*b,b,true},{a^-63,a^63,<ident\
ity ...>,true},{a^-63,a^-45*b,a^-108*b,true},{b,<identity ...>,b,true},{b,a^-36,b*a^-36,true},{b,a^72\
,b*a^72,true},{b,a^36*b,b*a^36*b,true},{b,a^72*b,b*a^72*b,true},{b,a^36,b*a^36,true},{b,a^-36*b,b*a^-\
36*b,true},{a^-54,a^54*b,b,true},{a^-54,a^54,<identity ...>,true},{a^-54,a^18,a^-36,true},{a^-54,a^-5\
4*b,a^-108*b,true},{a^-54,a^-18*b,a^-72*b,true},{a^-54,a^-18,a^-72,true},{a^-54,a^18*b,a^-36*b,true},\
{a^-36,<identity ...>,a^-36,true},{a^-36,b,a^-36*b,true},{a^-36,a^72,a^36,true},{a^-36,a^36*b,b,true}\
,{a^-36,a^72*b,a^36*b,true},{a^-36,a^36,<identity ...>,true},{a^-36,a^-36*b,a^-72*b,true},{a^72,<iden\
tity ...>,a^72,true},{a^72,b,a^72*b,true},{a^72,a^-36,a^36,true},{a^72,a^36*b,a^108*b,true},{a^72,a^7\
2*b,a^144*b,true},{a^72,a^36,a^108,true},{a^72,a^-36*b,a^36*b,true},{a^-63*b,a^-63,a^-63*b*a^-63,true\
},{a^-63*b,a^45,a^-63*b*a^45,true},{a^-63*b,a^9,a^-63*b*a^9,true},{a^-63*b,a^-27*b,a^-63*b*a^-27*b,tr\
ue},{a^-63*b,a^9*b,a^-63*b*a^9*b,true},{a^-63*b,a^-27,a^-63*b*a^-27,true},{a^-63*b,a^45*b,a^-63*b*a^4\
5*b,true},{a^27,a^-63,a^-36,true},{a^27,a^-63*b,a^-36*b,true},{a^27,a^45,a^72,true},{a^27,a^9,a^36,tr\
ue},{a^27,a^-27*b,b,true},{a^27,a^9*b,a^36*b,true},{a^27,a^-27,<identity ...>,true},{a^27,a^45*b,a^72\
*b,true},{a^45,a^27,a^72,true},{a^45,a^-9*b,a^36*b,true},{a^45,a^-9,a^36,true},{a^45,a^-45,<identity \
...>,true},{a^45,a^27*b,a^72*b,true},{a^45,a^63*b,a^108*b,true},{a^45,a^63,a^108,true},{a^45,a^-45*b,\
b,true},{a^9,a^27,a^36,true},{a^9,a^-9*b,b,true},{a^9,a^-9,<identity ...>,true},{a^9,a^-45,a^-36,true\
},{a^9,a^27*b,a^36*b,true},{a^9,a^63*b,a^72*b,true},{a^9,a^63,a^72,true},{a^9,a^-45*b,a^-36*b,true},{\
a^54*b,a^-54,a^54*b*a^-54,true},{a^54*b,a^54,a^54*b*a^54,true},{a^54*b,a^18,a^54*b*a^18,true},{a^54*b\
,a^-54*b,a^54*b*a^-54*b,true},{a^54*b,a^-18*b,a^54*b*a^-18*b,true},{a^54*b,a^-18,a^54*b*a^-18,true},{\
a^54*b,a^18*b,a^54*b*a^18*b,true},{a^36*b,<identity ...>,a^36*b,true},{a^36*b,b,a^36*b^2,true},{a^36*\
b,a^-36,a^36*b*a^-36,true},{a^36*b,a^72,a^36*b*a^72,true},{a^36*b,a^72*b,a^36*b*a^72*b,true},{a^36*b,\
a^36,a^36*b*a^36,true},{a^36*b,a^-36*b,a^36*b*a^-36*b,true},{a^72*b,<identity ...>,a^72*b,true},{a^72\
*b,b,a^72*b^2,true},{a^72*b,a^-36,a^72*b*a^-36,true},{a^72*b,a^72,a^72*b*a^72,true},{a^72*b,a^36*b,a^\
72*b*a^36*b,true},{a^72*b,a^36,a^72*b*a^36,true},{a^72*b,a^-36*b,a^72*b*a^-36*b,true},{a^54,a^-54,<id\
entity ...>,true},{a^54,a^54*b,a^108*b,true},{a^54,a^18,a^72,true},{a^54,a^-54*b,b,true},{a^54,a^-18*\
b,a^36*b,true},{a^54,a^-18,a^36,true},{a^54,a^18*b,a^72*b,true},{a^18,a^-54,a^-36,true},{a^18,a^54*b,\
a^72*b,true},{a^18,a^54,a^72,true},{a^18,a^-54*b,a^-36*b,true},{a^18,a^-18*b,b,true},{a^18,a^-18,<ide\
ntity ...>,true},{a^18,a^18*b,a^36*b,true},{a^36,<identity ...>,a^36,true},{a^36,b,a^36*b,true},{a^36\
,a^-36,<identity ...>,true},{a^36,a^72,a^108,true},{a^36,a^36*b,a^72*b,true},{a^36,a^72*b,a^108*b,tru\
e},{a^36,a^-36*b,b,true},{a^-9*b,a^27,a^-9*b*a^27,true},{a^-9*b,a^-9,a^-9*b*a^-9,true},{a^-9*b,a^-45,\
a^-9*b*a^-45,true},{a^-9*b,a^27*b,a^-9*b*a^27*b,true},{a^-9*b,a^63*b,a^-9*b*a^63*b,true},{a^-9*b,a^63\
,a^-9*b*a^63,true},{a^-9*b,a^-45*b,a^-9*b*a^-45*b,true},{a^-27*b,a^-63,a^-27*b*a^-63,true},{a^-27*b,a\
^-63*b,a^-27*b*a^-63*b,true},{a^-27*b,a^45,a^-27*b*a^45,true},{a^-27*b,a^9,a^-27*b*a^9,true},{a^-27*b\
,a^9*b,a^-27*b*a^9*b,true},{a^-27*b,a^-27,a^-27*b*a^-27,true},{a^-27*b,a^45*b,a^-27*b*a^45*b,true},{a\
^9*b,a^-63,a^9*b*a^-63,true},{a^9*b,a^-63*b,a^9*b*a^-63*b,true},{a^9*b,a^45,a^9*b*a^45,true},{a^9*b,a\
^9,a^9*b*a^9,true},{a^9*b,a^-27*b,a^9*b*a^-27*b,true},{a^9*b,a^-27,a^9*b*a^-27,true},{a^9*b,a^45*b,a^\
9*b*a^45*b,true},{a^-9,a^-63,a^-72,true},{a^-9,a^-63*b,a^-72*b,true},{a^-9,a^45,a^36,true},{a^-9,a^9,\
<identity ...>,true},{a^-9,a^-27*b,a^-36*b,true},{a^-9,a^9*b,b,true},{a^-9,a^-27,a^-36,true},{a^-9,a^\
45*b,a^36*b,true},{a^-45,a^-63,a^-108,true},{a^-45,a^-63*b,a^-108*b,true},{a^-45,a^45,<identity ...>,\
true},{a^-45,a^9,a^-36,true},{a^-45,a^-27*b,a^-72*b,true},{a^-45,a^9*b,a^-36*b,true},{a^-45,a^-27,a^-\
72,true},{a^-45,a^45*b,b,true},{a^-27,a^27,<identity ...>,true},{a^-27,a^-9*b,a^-36*b,true},{a^-27,a^\
-9,a^-36,true},{a^-27,a^-45,a^-72,true},{a^-27,a^27*b,b,true},{a^-27,a^63*b,a^36*b,true},{a^-27,a^63,\
a^36,true},{a^-27,a^-45*b,a^-72*b,true},{a^-54*b,a^-54,a^-54*b*a^-54,true},{a^-54*b,a^54*b,a^-54*b*a^\
54*b,true},{a^-54*b,a^54,a^-54*b*a^54,true},{a^-54*b,a^18,a^-54*b*a^18,true},{a^-54*b,a^-18*b,a^-54*b\
*a^-18*b,true},{a^-54*b,a^-18,a^-54*b*a^-18,true},{a^-54*b,a^18*b,a^-54*b*a^18*b,true},{a^-18*b,a^-54\
,a^-18*b*a^-54,true},{a^-18*b,a^54*b,a^-18*b*a^54*b,true},{a^-18*b,a^54,a^-18*b*a^54,true},{a^-18*b,a\
^18,a^-18*b*a^18,true},{a^-18*b,a^-54*b,a^-18*b*a^-54*b,true},{a^-18*b,a^-18,a^-18*b*a^-18,true},{a^-\
18*b,a^18*b,a^-18*b*a^18*b,true},{a^-36*b,<identity ...>,a^-36*b,true},{a^-36*b,b,a^-36*b^2,true},{a^\
-36*b,a^-36,a^-36*b*a^-36,true},{a^-36*b,a^72,a^-36*b*a^72,true},{a^-36*b,a^36*b,a^-36*b*a^36*b,true}\
,{a^-36*b,a^72*b,a^-36*b*a^72*b,true},{a^-36*b,a^36,a^-36*b*a^36,true},{a^-18,a^-54,a^-72,true},{a^-1\
8,a^54*b,a^36*b,true},{a^-18,a^54,a^36,true},{a^-18,a^18,<identity ...>,true},{a^-18,a^-54*b,a^-72*b,\
true},{a^-18,a^-18*b,a^-36*b,true},{a^-18,a^18*b,b,true},{a^27*b,a^27,a^27*b*a^27,true},{a^27*b,a^-9*\
b,a^27*b*a^-9*b,true},{a^27*b,a^-9,a^27*b*a^-9,true},{a^27*b,a^-45,a^27*b*a^-45,true},{a^27*b,a^63*b,\
a^27*b*a^63*b,true},{a^27*b,a^63,a^27*b*a^63,true},{a^27*b,a^-45*b,a^27*b*a^-45*b,true},{a^63*b,a^27,\
a^63*b*a^27,true},{a^63*b,a^-9*b,a^63*b*a^-9*b,true},{a^63*b,a^-9,a^63*b*a^-9,true},{a^63*b,a^-45,a^6\
3*b*a^-45,true},{a^63*b,a^27*b,a^63*b*a^27*b,true},{a^63*b,a^63,a^63*b*a^63,true},{a^63*b,a^-45*b,a^6\
3*b*a^-45*b,true},{a^45*b,a^-63,a^45*b*a^-63,true},{a^45*b,a^-63*b,a^45*b*a^-63*b,true},{a^45*b,a^45,\
a^45*b*a^45,true},{a^45*b,a^9,a^45*b*a^9,true},{a^45*b,a^-27*b,a^45*b*a^-27*b,true},{a^45*b,a^9*b,a^4\
5*b*a^9*b,true},{a^45*b,a^-27,a^45*b*a^-27,true},{a^63,a^-63,<identity ...>,true},{a^63,a^-63*b,b,tru\
e},{a^63,a^45,a^108,true},{a^63,a^9,a^72,true},{a^63,a^-27*b,a^36*b,true},{a^63,a^9*b,a^72*b,true},{a\
^63,a^-27,a^36,true},{a^63,a^45*b,a^108*b,true},{a^18*b,a^-54,a^18*b*a^-54,true},{a^18*b,a^54*b,a^18*\
b*a^54*b,true},{a^18*b,a^54,a^18*b*a^54,true},{a^18*b,a^18,a^18*b*a^18,true},{a^18*b,a^-54*b,a^18*b*a\
^-54*b,true},{a^18*b,a^-18*b,a^18*b*a^-18*b,true},{a^18*b,a^-18,a^18*b*a^-18,true},{a^-45*b,a^27,a^-4\
5*b*a^27,true},{a^-45*b,a^-9*b,a^-45*b*a^-9*b,true},{a^-45*b,a^-9,a^-45*b*a^-9,true},{a^-45*b,a^-45,a\
^-45*b*a^-45,true},{a^-45*b,a^27*b,a^-45*b*a^27*b,true},{a^-45*b,a^63*b,a^-45*b*a^63*b,true},{a^-45*b\
,a^63,a^-45*b*a^63,true},
```

Figure 9: GAP Coding to Determine the Direction of Elements in Subgroup $\langle b, a^{12}, a^8 b\rangle$ of QD_{32}

By Definition 3,

$1 \to b$ since $1 \cdot b = b \in H_3$

$1 \to a^{12}$ since $1 \cdot a^{12} = a^{12} \in H_3$

$1 \to a^{8}$ since $1 \cdot a^{8} = a^{8} \in H_3$

$a^7 b \to a^{11}$ since $a^7 b \cdot a^{11} = a^4 b \in H_3$

$a^7 b \to a^{7}$ since $a^7 b \cdot a^{7} = a^8 b \in H_3$

$a^7 b \to a^{3}$ since $a^7 b \cdot a^{3} = a^{12} b \in H_3$

$1 \to a^4b$ since $1 \cdot a^4b = a^4b \in H_3$

$1 \to a^8b$ since $1 \cdot a^8b = a^8b \in H_3$

$1 \to a^4$ since $1 \cdot a^4 = a^4 \in H_3$

$1 \to a^{12}b$ since $1 \cdot a^{12}b = a^{12}b \in H_3$

$a \to a^{11}$ since $a \cdot a^{11} = a^{12} \in H_3$

$a \to a^7b$ since $a \cdot a^7b = a^8b \in H_3$

$a \to a^7$ since $a \cdot a^7 = a^8 \in H_3$

$a \to a^3$ since $a \cdot a^3 = a^4 \in H_3$

$a \to a^{11}b$ since $a \cdot a^{11}b = a^{12}b \in H_3$

$a \to a^{15}b$ since $a \cdot a^{15}b = b \in H_3$

$a \to a^{15}$ since $a \cdot a^{15} = 1 \in H_3$

$a \to a^3b$ since $a \cdot a^3b = a^4b \in H_3$

$b \to 1$ since $b \cdot 1 = b \in H_3$

$b \to a^{12}$ since $b \cdot a^{12} = a^4b \in H_3$

$b \to a^8$ since $b \cdot a^8 = a^8b \in H_3$

$b \to a^4b$ since $b \cdot a^4b = a^{12} \in H_3$

$b \to a^8b$ since $b \cdot a^8b = a^8 \in H_3$

$b \to a^4$ since $b \cdot a^4 = a^{12}b \in H_3$

$b \to a^{12}b$ since $b \cdot a^{12}b = a^4 \in H_3$

$a^{10} \to a^6b$ since $a^{10} \cdot a^6b = b \in H_3$

$a^{10} \to a^6$ since $a^{10} \cdot a^6 = 1 \in H_3$

$a^{10} \to a^2$ since $a^{10} \cdot a^2 = a^{12} \in H_3$

$a^{10} \to a^{10}b$ since $a^{10} \cdot a^{10}b = a^4b \in H_3$

$a^{10} \to a^{14}b$ since $a^{10} \cdot a^{14}b = a^8b \in H_3$

$a^{10} \to a^{14}$ since $a^{10} \cdot a^{14} = a^8 \in H_3$

$a^{10} \to a^2b$ since $a^{10} \cdot a^2b = a^{12}b \in H_3$

$a^{12} \to 1$ since $a^{12} \cdot 1 = a^{12} \in H_3$

$a^{12} \to b$ since $a^{12} \cdot b = a^{12}b \in H_3$

$a^{12} \to a^8$ since $a^{12} \cdot a^8 = a^4 \in H_3$

$a^{12} \to a^4b$ since $a^{12} \cdot a^4b = b \in H_3$

$a^{12} \to a^8b$ since $a^{12} \cdot a^8b = a^4b \in H_3$

$a^7b \to a^{11}b$ since $a^7b \cdot a^{11}b = a^4 \in H_3$

$a^7b \to a^{15}b$ since $a^7b \cdot a^{15}b = 1 \in H_3$

$a^7b \to a^{15}$ since $a^7b \cdot a^{15} = b \in H_3$

$a^7b \to a^3b$ since $a^7b \cdot a^3b = a^{12} \in H_3$

$a^5b \to a$ since $a^5b \cdot a = a^{12}b \in H_3$

$a^5b \to ab$ since $a^5b \cdot ab = a^{12} \in H_3$

$a^5b \to a^{13}$ since $a^5b \cdot a^{13} = b \in H_3$

$a^5b \to a^9$ since $a^5b \cdot a^9 = a^4b \in H_3$

$a^5b \to a^9b$ since $a^5b \cdot a^9b = a^4 \in H_3$

$a^5b \to a^5$ since $a^5b \cdot a^5 = a^8b \in H_3$

$a^5b \to a^{13}b$ since $a^5b \cdot a^{13}b = 1 \in H_3$

$a^9b \to a$ since $a^9b \cdot a = b \in H_3$

$a^9b \to ab$ since $a^9b \cdot ab = 1 \in H_3$

$a^9b \to a^{13}$ since $a^9b \cdot a^{13} = a^4b \in H_3$

$a^9b \to a^9$ since $a^9b \cdot a^9 = a^8b \in H_3$

$a^9b \to a^5b$ since $a^9b \cdot a^5b = a^{12} \in H_3$

$a^9b \to a^5$ since $a^9b \cdot a^5 = a^{12}b \in H_3$

$a^9b \to a^{13}b$ since $a^9b \cdot a^{13}b = a^4 \in H_3$

$a^7 \to a$ since $a^7 \cdot a = a^8 \in H_3$

$a^7 \to ab$ since $a^7 \cdot ab = a^8b \in H_3$

$a^7 \to a^{13}$ since $a^7 \cdot a^{13} = a^4 \in H_3$

$a^7 \to a^9$ since $a^7 \cdot a^9 = 1 \in H_3$

$a^7 \to a^5b$ since $a^7 \cdot a^5b = a^{12}b \in H_3$

$a^7 \to a^9b$ since $a^7 \cdot a^9b = b \in H_3$

$a^7 \to a^5$ since $a^7 \cdot a^5 = a^{12} \in H_3$

$a^7 \to a^{13}b$ since $a^7 \cdot a^{13}b = a^4b \in H_3$

$a^3 \to a$ since $a^3 \cdot a = a^4 \in H_3$

$a^3 \to ab$ since $a^3 \cdot ab = a^4b \in H_3$

$a^3 \to a^{13}$ since $a^3 \cdot a^{13} = 1 \in H_3$

$a^3 \to a^9$ since $a^3 \cdot a^9 = a^{12} \in H_3$

$a^3 \to a^5b$ since $a^3 \cdot a^5b = a^8b \in H_3$

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$a^{12} \to a^4$ since $a^{12} \cdot a^4 = 1 \in H_3$

$a^{12} \to a^{12}b$ since $a^{12} \cdot a^{12}b = a^8b \in H_3$

$a^8 \to 1$ since $a^8 \cdot 1 = a^8 \in H_3$

$a^8 \to b$ since $a^8 \cdot b = a^8b \in H_3$

$a^8 \to a^{12}$ since $a^8 \cdot a^{12} = a^4 \in H_3$

$a^8 \to a^4b$ since $a^8 \cdot a^4b = a^{12}b \in H_3$

$a^8 \to a^8b$ since $a^8 \cdot a^8b = b \in H_3$

$a^8 \to a^4$ since $a^8 \cdot a^4 = a^{12} \in H_3$

$a^8 \to a^{12}b$ since $a^8 \cdot a^{12}b = a^4b \in H_3$

$ab \to a$ since $ab \cdot a = a^8b \in H_3$

$ab \to a^{13}$ since $ab \cdot a^{13} = a^{12}b \in H_3$

$ab \to a^9$ since $ab \cdot a^9 = b \in H_3$

$ab \to a^5b$ since $ab \cdot a^5b = a^4 \in H_3$

$ab \to a^9b$ since ab $\cdot a^9b = 1 \in H_3$

$ab \to a^5$ since $ab \cdot a^5 = a^4b \in H_3$

$ab \to a^{13}b$ since $ab \cdot a^{13}b = a^{12} \in H_3$

$a^{11} \to a$ since $a^{11} \cdot a = a^{12} \in H_3$

$a^{11} \to ab$ since $a^{11} \cdot ab = a^{12}b \in H_3$

$a^{11} \to a^{13}$ since $a^{11} \cdot a^{13} = a^8 \in H_3$

$a^{11} \to a^9$ since $a^{11} \cdot a^9 = a^4 \in H_3$

$a^{11} \to a^5b$ since $a^{11} \cdot a^5b = b \in H_3$

$a^{11} \to a^9b$ since $a^{11} \cdot a^9b = a^4b \in H_3$

$a^{11} \to a^5$ since $a^{11} \cdot a^5 = 1 \in H_3$

$a^{11} \to a^{13}b$ since $a^{11} \cdot a^{13}b = a^8b \in H_3$

$a^{13} \to a^{11}$ since $a^{13} \cdot a^{11} = a^8 \in H_3$

$a^{13} \to a^7b$ since $a^{13} \cdot a^7b = a^4b \in H_3$

$a^{13} \to a^7$ since $a^{13} \cdot a^7 = a^4 \in H_3$

$a^{13} \to a^3$ since $a^{13} \cdot a^3 = 1 \in H_3$

$a^{13} \to a^{11}b$ since $a^{13} \cdot a^{11}b = a^8b \in H_3$

$a^{13} \to a^{15}b$ since $a^{13} \cdot a^{15}b = a^{12}b \in H_3$

$a^{13} \to a^{15}$ since $a^{13} \cdot a^{15} = b \in H_3$

$a^3 \to a^9b$ since $a^3 \cdot a^9b = a^{12}b \in H_3$

$a^3 \to a^5$ since $a^3 \cdot a^5 = a^8 \in H_3$

$a^3 \to a^{13}b$ since $a^3 \cdot a^{13}b = b \in H_3$

$a^5 \to a^{11}$ since $a^5 \cdot a^{11} = 1 \in H_3$

$a^5 \to a^7b$ since $a^5 \cdot a^7b = a^{12}b \in H_3$

$a^5 \to a^7$ since $a^5 \cdot a^7 = a^{12} \in H_3$

$a^5 \to a^3$ since $a^5 \cdot a^3 = a^8 \in H_3$

$a^5 \to a^{11}b$ since $a^5 \cdot a^{11}b = b \in H_3$

$a^5 \to a^{15}b$ since $a^5 \cdot a^{15}b = a^4b \in H_3$

$a^5 \to a^{15}$ since $a^5 \cdot a^{15} = a^4 \in H_3$

$a^5 \to a^3b$ since $a^5 \cdot a^3b = a^8b \in H_3$

$a^{10}b \to a^{10}$ since $a^{10}b \cdot a^{10} = b \in H_3$

$a^{10}b \to a^6b$ since $a^{10}b \cdot a^6b = a^4 \in H_3$

$a^{10}b \to a^6$ since $a^{10}b \cdot a^6 = a^4b \in H_3$

$a^{10}b \to a^2$ since $a^{10}b \cdot a^2 = a^8b \in H_3$

$a^{10}b \to a^{14}b$ since $a^{10}b \cdot a^{14}b = a^{12} \in H_3$

$a^{10}b \to a^{14}$ since $a^{10}b \cdot a^{14} = a^{12}b \in H_3$

$a^{10}b \to a^2b$ since $a^{10}b \cdot a^2b = a^8 \in H_3$

$a^{14}b \to a^{10}$ since $a^{14}b \cdot a^{10} = a^4b \in H_3$

$a^{14}b \to a^6b$ since $a^{14}b \cdot a^6b = a^8 \in H_3$

$a^{14}b \to a^6$ since $a^{14}b \cdot a^6 = a^8b \in H_3$

$a^{14}b \to a^2$ since $a^{14}b \cdot a^2 = a^{12}b \in H_3$

$a^{14}b \to a^{10}b$ since $a^{14}b \cdot a^{10}b = a^4 \in H_3$

$a^{14}b \to a^{14}$ since $a^{14}b \cdot a^{14} = b \in H_3$

$a^{14}b \to a^2b$ since $a^{14}b \cdot a^2b = a^{12} \in H_3$

$a^{12}b \to 1$ since $a^{12}b \cdot 1 = a^{12}b \in H_3$

$a^{12}b \to b$ since $a^{12}b \cdot b = a^{12} \in H_3$

$a^{12}b \to a^{12}$ since $a^{12}b \cdot a^{12} = b \in H_3$

$a^{12}b \to a^8$ since $a^{12}b \cdot a^8 = a^4b \in H_3$

$a^{12}b \to a^4b$ since $a^{12}b \cdot a^4b = a^8 \in H_3$

$a^{12}b \to a^8b$ since $a^{12}b \cdot a^8b = a^4 \in H_3$

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$a^{13} \to a^3b$ since $a^{13} \cdot a^3b = b \in H_3$
$a^9 \to a^{11}$ since $a^9 \cdot a^{11} = a^4 \in H_3$
$a^9 \to a^7b$ since $a^9 \cdot a^7b = b \in H_3$
$a^9 \to a^7$ since $a^9 \cdot a^7 = 1 \in H_3$
$a^9 \to a^3$ since $a^9 \cdot a^3 = a^{12} \in H_3$
$a^9 \to a^{11}b$ since $a^9 \cdot a^{11}b = a^4b \in H_3$
$a^9 \to a^{15}b$ since $a^9 \cdot a^{15}b = a^8b \in H_3$
$a^9 \to a^{15}$ since $a^9 \cdot a^{15} = a^8 \in H_3$
$a^9 \to a^3b$ since $a^9 \cdot a^3b = a^{12}b \in H_3$
$a^6b \to a^{10}$ since $a^6b \cdot a^{10} = a^{12}b \in H_3$
$a^6b \to a^6$ since $a^6b \cdot a^6 = b \in H_3$
$a^6b \to a^2$ since $a^6b \cdot a^2 = a^4b \in H_3$
$a^6b \to a^{10}b$ since $a^6b \cdot a^{10}b = a^{12} \in H_3$
$a^6b \to a^{14}b$ since $a^6b \cdot a^{14}b = a^8 \in H_3$
$a^6b \to a^{14}$ since $a^6b \cdot a^{14} = a^8b \in H_3$
$a^6b \to a^2b$ since $a^6b \cdot a^2b = a^4 \in H_3$
$a^4b \to 1$ since $a^4b \cdot 1 = a^4b \in H_3$
$a^4b \to b$ since $a^4b \cdot b = a^4 \in H_3$
$a^4b \to a^{12}$ since $a^4b \cdot a^{12} = a^8b \in H_3$
$a^4b \to a^8$ since $a^4b \cdot a^8 = a^{12}b \in H_3$
$a^4b \to a^8b$ since $a^4b \cdot a^8b = a^{12} \in H_3$
$a^4b \to a^4$ since $a^4b \cdot a^4 = b \in H_3$
$a^4b \to a^{12}b$ since $a^4b \cdot a^{12}b = a^8 \in H_3$
$a^8b \to 1$ since $a^8b \cdot 1 = a^8b \in H_3$
$a^8b \to b$ since $a^8b \cdot b = a^8 \in H_3$
$a^8b \to a^{12}$ since $a^8b \cdot a^{12} = a^{12}b \in H_3$
$a^8b \to a^8$ since $a^8b \cdot a^8 = b \in H_3$
$a^8b \to a^4b$ since $a^8b \cdot a^4b = a^4 \in H_3$
$a^8b \to a^4$ since $a^8b \cdot a^4 = a^4b \in H_3$
$a^8b \to a^{12}b$ since $a^8b \cdot a^{12}b = a^{12} \in H_3$
$a^6 \to a^{10}$ since $a^6 \cdot a^{10} = a^{16} \in H_3$

$a^{12}b \to a^4$ since $a^{12}b \cdot a^4 = a^8b \in H_3$
$a^{14} \to a^{10}$ since $a^{14} \cdot a^{10} = a^8 \in H_3$
$a^{14} \to a^6b$ since $a^{14} \cdot a^6b = a^4b \in H_3$
$a^{14} \to a^6$ since $a^{14} \cdot a^6 = a^4 \in H_3$
$a^{14} \to a^2$ since $a^{14} \cdot a^2 = 1 \in H_3$
$a^{14} \to a^{10}b$ since $a^{14} \cdot a^{10}b = a^8b \in H_3$
$a^{14} \to a^{14}b$ since $a^{14} \cdot a^{14}b = a^{12}b \in H_3$
$a^{14} \to a^2b$ since $a^{14} \cdot a^2b = b \in H_3$
$a^{11}b \to a^{11}$ since $a^{11}b \cdot a^{11} = a^8b \in H_3$
$a^{11}b \to a^7b$ since $a^{11}b \cdot a^7b = a^{12} \in H_3$
$a^{11}b \to a^7$ since $a^{11}b \cdot a^7 = a^{12}b \in H_3$
$a^{11}b \to a^3$ since $a^{11}b \cdot a^3 = b \in H_3$
$a^{11}b \to a^{15}b$ since $a^{11}b \cdot a^{15}b = a^4 \in H_3$
$a^{11}b \to a^{15}$ since $a^{11}b \cdot a^{15} = a^4b \in H_3$
$a^{11}b \to a^3b$ since $a^{11}b \cdot a^3b = 1 \in H_3$
$a^{15}b \to a^{11}$ since $a^{15}b \cdot a^{11} = a^{12}b \in H_3$
$a^{15}b \to a^7b$ since $a^{15}b \cdot a^7b = 1 \in H_3$
$a^{15}b \to a^7$ since $a^{15}b \cdot a^7 = b \in H_3$
$a^{15}b \to a^3$ since $a^{15}b \cdot a^3 = a^4b \in H_3$
$a^{15}b \to a^{11}b$ since $a^{15}b \cdot a^{11}b = a^{12} \in H_3$
$a^{15}b \to a^{15}$ since $a^{15}b \cdot a^{15} = a^8b \in H_3$
$a^{15}b \to a^3b$ since $a^{15}b \cdot a^3b = a^4 \in H_3$
$a^{13}b \to a$ since $a^{13}b \cdot a = a^4b \in H_3$
$a^{13}b \to ab$ since $a^{13}b \cdot ab = a^4 \in H_3$
$a^{13}b \to a^{13}$ since $a^{13}b \cdot a^{13} = a^8b \in H_3$
$a^{13}b \to a^9$ since $a^{13}b \cdot a^9 = a^{12}b \in H_3$
$a^{13}b \to a^5b$ since $a^{13}b \cdot a^5b = 1 \in H_3$
$a^{13}b \to a^9b$ since $a^{13}b \cdot a^9b = a^{12} \in H_3$
$a^{13}b \to a^5$ since $a^{13}b \cdot a^5 = b \in H_3$
$a^{15} \to a$ since $a^{15} \cdot a = 1 \in H_3$
$a^{15} \to ab$ since $a^{15} \cdot ab = b \in H_3$

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$a^6 \to a^6b$ since $a^6 \cdot a^6b = a^{12}b \in H_3$
$a^6 \to a^2$ since $a^6 \cdot a^2 = a^8 \in H_3$
$a^6 \to a^{10}b$ since $a^6 \cdot a^{10}b = b \in H_3$
$a^6 \to a^{14}b$ since $a^6 \cdot a^{14}b = a^4b \in H_3$
$a^6 \to a^{14}$ since $a^6 \cdot a^{14} = a^4 \in H_3$
$a^6 \to a^2b$ since $a^6 \cdot a^2b = a^8b \in H_3$
$a^2 \to a^{10}$ since $a^2 \cdot a^{10} = a^{12} \in H_3$
$a^2 \to a^6b$ since $a^2 \cdot a^6b = a^8b \in H_3$
$a^2 \to a^6$ since $a^2 \cdot a^6 = a^8 \in H_3$
$a^2 \to a^{10}b$ since $a^2 \cdot a^{10}b = a^{12}b \in H_3$
$a^2 \to a^{14}b$ since $a^2 \cdot a^{14}b = b \in H_3$
$a^2 \to a^{14}$ since $a^2 \cdot a^{14} = 1 \in H_3$
$a^2 \to a^2b$ since $a^2 \cdot a^2b = a^4b \in H_3$
$a^4 \to 1$ since $a^4 \cdot 1 = a^4 \in H_3$
$a^4 \to b$ since $a^4 \cdot b = a^4b \in H_3$
$a^4 \to a^{12}$ since $a^4 \cdot a^{12} = 1 \in H_3$
$a^4 \to a^8$ since $a^4 \cdot a^8 = a^{12} \in H_3$
$a^4 \to a^4b$ since $a^4 \cdot a^4b = a^8b \in H_3$
$a^4 \to a^8b$ since $a^4 \cdot a^8b = a^{12}b \in H_3$
$a^4 \to a^{12}b$ since $a^4 \cdot a^{12}b = b \in H_3$

$a^{15} \to a^{13}$ since $a^{15} \cdot a^{13} = a^{12} \in H_3$
$a^{15} \to a^9$ since $a^{15} \cdot a^9 = a^8 \in H_3$
$a^{15} \to a^5b$ since $a^{15} \cdot a^5b = a^4b \in H_3$
$a^{15} \to a^9b$ since $a^{15} \cdot a^9b = a^8b \in H_3$
$a^{15} \to a^5$ since $a^{15} \cdot a^5 = a^4 \in H_3$
$a^{15} \to a^{13}b$ since $a^{15} \cdot a^{13}b = a^{12}b \in H_3$
$a^2b \to a^{10}$ since $a^2b \cdot a^{10} = a^8b \in H_3$
$a^2b \to a^6b$ since $a^2b \cdot a^6b = a^{12} \in H_3$
$a^2b \to a^6$ since $a^2b \cdot a^6 = a^{12}b \in H_3$
$a^2b \to a^2$ since $a^2b \cdot a^2 = b \in H_3$
$a^2b \to a^{10}b$ since $a^2b \cdot a^{10}b = a^8 \in H_3$
$a^2b \to a^{14}b$ since $a^2b \cdot a^{14}b = a^4 \in H_3$
$a^2b \to a^{14}$ since $a^2b \cdot a^{14} = a^4b \in H_3$
$a^3b \to a^{11}$ since $a^3b \cdot a^{11} = b \in H_3$
$a^3b \to a^7b$ since $a^3b \cdot a^7b = a^4 \in H_3$
$a^3b \to a^7$ since $a^3b \cdot a^7 = a^4b \in H_3$
$a^3b \to a^3$ since $a^3b \cdot a^3 = a^8b \in H_3$
$a^3b \to a^{11}b$ since $a^3b \cdot a^{11}b = 1 \in H_3$
$a^3b \to a^{15}b$ since $a^3b \cdot a^{15}b = a^{12} \in H_3$
$a^3b \to a^{15}$ since $a^3b \cdot a^{15} = a^{12}b \in H_3$

Therefore, $\Gamma_{H_3}^{NN}(QD_{32})$ is the graph shown in Figure 8.

Proposition 4, Let $\Gamma_{H_4}^{NN}(QD_{32})$ be the non-normal cyclic subgroup graph of $QD_{32}, H_4 = \langle a^{14}b, a^6b, a^8, a^{12} \rangle = \{1, a^4, a^8, a^{12}, b, a^4b, a^8b, a^{12}b\}$. Then, $\Gamma_{H_4}^{NN}(QD_{32})$ is as shown in Figure 10.

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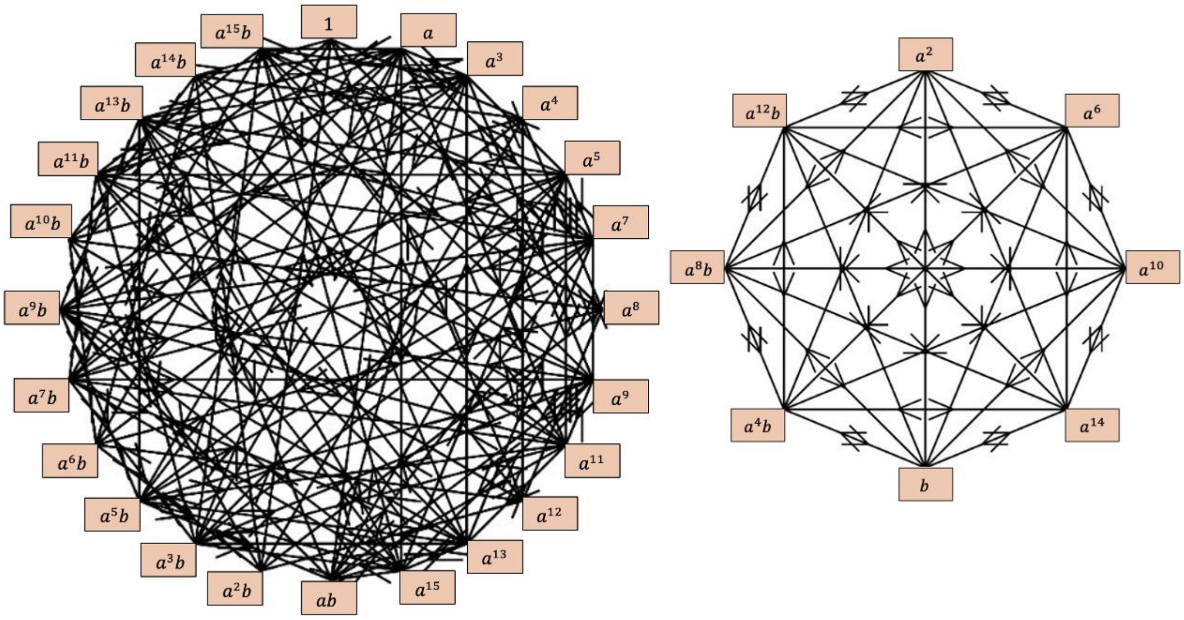

Figure 10: $\Gamma_{H_4}^{NN}(QD_{32})$

Proof. Let $H_4 = \langle a^{14}b, a^6b, a^8, a^{12}\rangle = \begin{Bmatrix} 1, a^4, a^6, a^8, a^{12}, a^{14}, b, a^4b, \\ a^5b, a^6b, a^8b, a^{12}b, a^{14}b \end{Bmatrix}$. The directions of every elements in subgroups are determined by definition and GAP software as shown in Figure 11.

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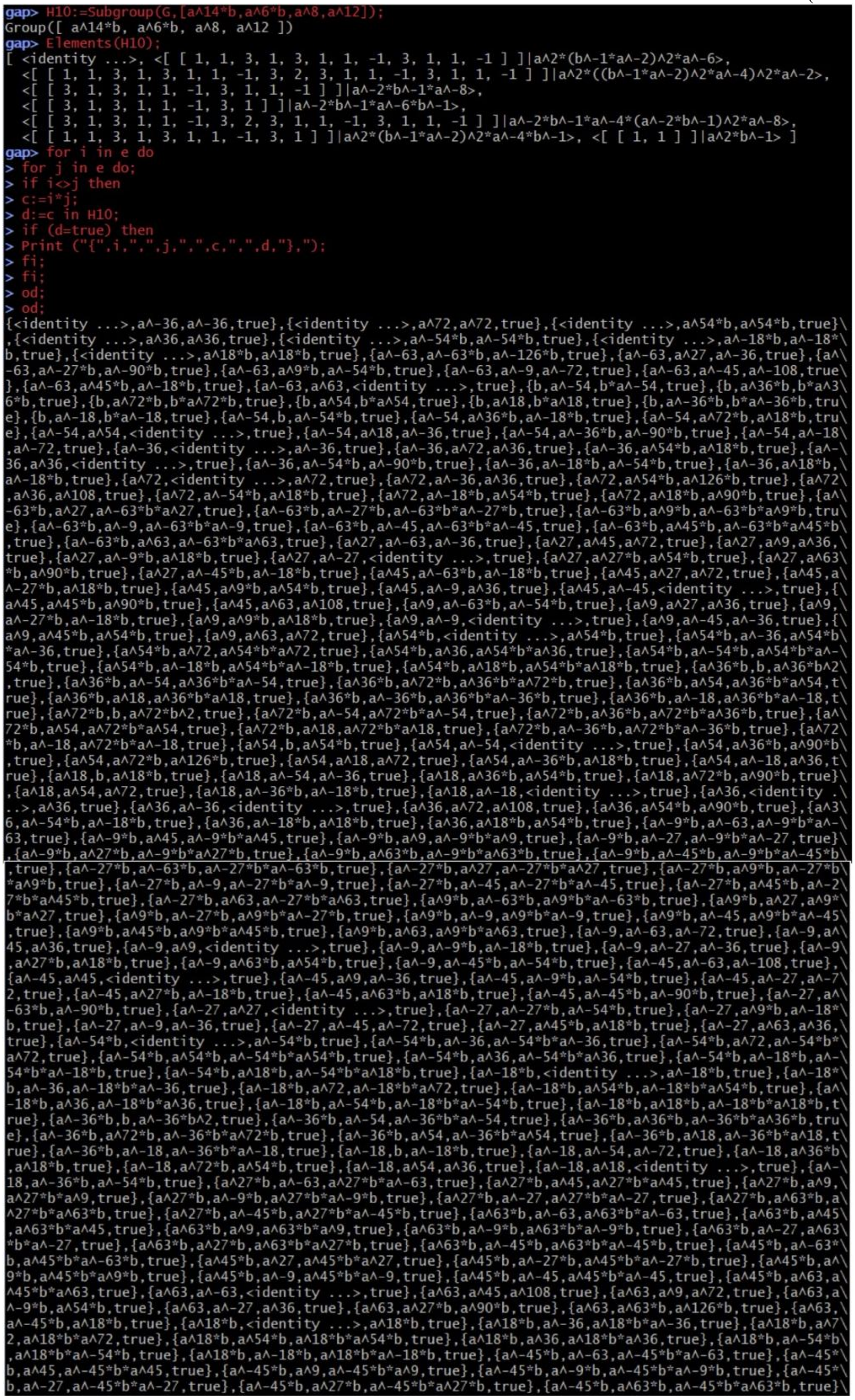

Figure 11: GAP coding to determine the direction of elements in subgroup $\langle a^{14}b, a^{6}b, a^{8}, a^{12}\rangle$ of QD_{32}

By Definition 3,

$a^{15} \to a$ since $a^{15} \cdot a = 1 \in H_4$

$a \to a^{15}$ since $a \cdot a^{15} = 1 \in H_4$

$a^{14} \to a^{2}$ since $a^{14} \cdot a^{2} = 1 \in H_4$

$a^{9} \to a^{3}$ since $a^{9} \cdot a^{3} = a^{12} \in H_4$

$a^{3} \to a^{9}$ since $a^{3} \cdot a^{9} = a^{12} \in H_4$

$a^{7} \to a^{5}$ since $a^{7} \cdot a^{5} = a^{12} \in H_4$

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$a^2 \to a^{14}$ since $a^2 \cdot a^{14} = 1 \in H_4$
$a^{13} \to a^3$ since $a^{13} \cdot a^3 = 1 \in H_4$
$a^3 \to a^{13}$ since $a^3 \cdot a^{13} = 1 \in H_4$
$a^{12} \to a^4$ since $a^{12} \cdot a^4 = 1 \in H_4$
$a^4 \to a^{12}$ since $a^4 \cdot a^{12} = 1 \in H_4$
$a^{11} \to a^5$ since $a^{11} \cdot a^5 = 1 \in H_4$
$a^5 \to a^{11}$ since $a^5 \cdot a^{11} = 1 \in H_4$
$a^{10} \to a^6$ since $a^{10} \cdot a^6 = 1 \in H_4$
$a^6 \to a^{10}$ since $a^6 \cdot a^{10} = 1 \in H_4$
$a^9 \to a^7$ since $a^9 \cdot a^7 = 1 \in H_4$
$a^7 \to a^9$ since $a^7 \cdot a^9 = 1 \in H_4$
$a^9b \to ab$ since $a^9b \cdot ab = 1 \in H_4$
$ab \to a^9b$ since $ab \cdot a^9b = 1 \in H_4$
$a^{11}b \to a^3b$ since $a^{11}b \cdot a^3b = 1 \in H_4$
$a^3b \to a^{11}b$ since $a^3b \cdot a^{11}b = 1 \in H_4$
$a^{13}b \to a^5b$ since $a^{13}b \cdot a^5b = 1 \in H_4$
$a^5b \to a^{13}b$ since $a^5b \cdot a^{13}b = 1 \in H_4$
$a^{15}b \to a^7b$ since $a^{15}b \cdot a^7b = 1 \in H_4$
$a^7b \to a^{15}b$ since $a^7b \cdot a^{15}b = 1 \in H_4$
$a^{14}b \to a$ since $a^{14}b \cdot a = a^5b \in H_4$
$1 \to a^{14}b$ since $1 \cdot a^{14}b = a^{14}b \in H_4$
$a^6b \to a^8$ since $a^6b \cdot a^8 = a^{14}b \in H_4$
$a^8 \to a^6b$ since $a^8 \cdot a^6b = a^{14}b \in H_4$
$a^7b \to a$ since $a^7b \cdot a = a^{14}b \in H_4$
$b \to a^2$ since $b \cdot a^2 = a^{14}b \in H_4$
$a^9b \to a^3$ since $a^9b \cdot a^3 = a^{14}b \in H_4$
$a^2b \to a^4$ since $a^2b \cdot a^4 = a^{14}b \in H_4$
$a^{11}b \to a^5$ since $a^{11}b \cdot a^5 = a^{14}b \in H_4$
$a^4b \to a^6$ since $a^4b \cdot a^6 = a^{14}b \in H_4$
$a^{13}b \to a^7$ since $a^{13}b \cdot a^7 = a^{14}b \in H_4$
$a^{15}b \to a^9$ since $a^{15}b \cdot a^9 = a^{14}b \in H_4$

$a^5 \to a^7$ since $a^5 \cdot a^7 = a^{12} \in H_4$
$a^{15} \to a^{13}$ since $a^{15} \cdot a^{13} = a^{12} \in H_4$
$a^{13} \to a^{15}$ since $a^{13} \cdot a^{15} = a^{12} \in H_4$
$a^{12}b \to b$ since $a^{12}b \cdot b = a^{12} \in H_4$
$a^5b \to ab$ since $a^5b \cdot ab = a^{12} \in H_4$
$a^{14}b \to a^2b$ since $a^{14}b \cdot a^2b = a^{12} \in H_4$
$a^7b \to a^3b$ since $a^7b \cdot a^3b = a^{12} \in H_4$
$b \to a^4b$ since $b \cdot a^4b = a^{12} \in H_4$
$a^9b \to a^5b$ since $a^9b \cdot a^5b = a^{12} \in H_4$
$a^2b \to a^6b$ since $a^2b \cdot a^6b = a^{12} \in H_4$
$a^{11}b \to a^7b$ since $a^{11}b \cdot a^7b = a^{12} \in H_4$
$a^4b \to a^8b$ since $a^4b \cdot a^8b = a^{12} \in H_4$
$a^{13}b \to a^9b$ since $a^{13}b \cdot a^9b = a^{12} \in H_4$
$a^6b \to a^{10}b$ since $a^6b \cdot a^{10}b = a^{12} \in H_4$
$a^{15}b \to a^{11}b$ since $a^{15}b \cdot a^{11}b = a^{12} \in H_4$
$a^8b \to a^{12}b$ since $a^8b \cdot a^{12}b = a^{12} \in H_4$
$ab \to a^{13}b$ since $ab \cdot a^{13}b = a^{12} \in H_4$
$a^9b \to a^{13}b$ since $a^9b \cdot a^{13}b = a^4 \in H_4$
$a^7 \to a$ since $a^7 \cdot a = a^8 \in H_4$
$a^7 \to ab$ since $a^7 \cdot ab = a^8b \in H_4$
$a^7 \to a^{13}$ since $a^7 \cdot a^{13} = a^4 \in H_4$
$a^7 \to a^9$ since $a^7 \cdot a^9 = 1 \in H_4$
$a^7 \to a^5b$ since $a^7 \cdot a^5b = a^{12}b \in H_4$
$a^7 \to a^9b$ since $a^7 \cdot a^9b = b \in H_4$
$a^7 \to a^5$ since $a^7 \cdot a^5 = a^{12} \in H_4$
$a^7 \to a^{13}b$ since $a^7 \cdot a^{13}b = a^4b \in H_4$
$a^3 \to a$ since $a^3 \cdot a = a^4 \in H_4$
$a^3 \to ab$ since $a^3 \cdot ab = a^4b \in H_4$
$a^3 \to a^{13}$ since $a^3 \cdot a^{13} = 1 \in H_4$
$a^3 \to a^9$ since $a^3 \cdot a^9 = a^{12} \in H_4$
$a^3 \to a^5b$ since $a^3 \cdot a^5b = a^8b \in H_4$

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$a^8b \to a^{10}$ since $a^8b \cdot a^{10} = a^{14}b \in H_4$
$ab \to a^{11}$ since $ab \cdot a^{11} = a^{14}b \in H_4$
$a^{10}b \to a^{12}$ since $a^{10}b \cdot a^{12} = a^{14}b \in H_4$
$a^3b \to a^{13}$ since $a^3b \cdot a^{13} = a^{14}b \in H_4$
$a^{12}b \to a^{14}$ since $a^{12}b \cdot a^{14} = a^{14}b \in H_4$
$a^5b \to a^{15}$ since $a^5b \cdot a^{15} = a^{14}b \in H_4$
$a^{14} \to b$ since $a^{14} \cdot b = a^{14}b \in H_4$
$a^{13} \to ab$ since $a^{13} \cdot ab = a^{14}b \in H_4$
$a^{12} \to a^2b$ since $a^{12} \cdot a^2b = a^{14}b \in H_4$
$a^{11} \to a^3b$ since $a^{11} \cdot a^3b = a^{14}b \in H_4$
$a^{10} \to a^4b$ since $a^{10} \cdot a^4b = a^{14}b \in H_4$
$a^9 \to a^5b$ since $a^9 \cdot a^5b = a^{14}b \in H_4$
$a^7 \to a^7b$ since $a^7 \cdot a^7b = a^{14}b \in H_4$
$a^6 \to a^8b$ since $a^6 \cdot a^8b = a^{14}b \in H_4$
$a^5 \to a^9b$ since $a^5 \cdot a^9b = a^{14}b \in H_4$
$a^4 \to a^{10}b$ since $a^4 \cdot a^{10}b = a^{14}b \in H_4$
$a^3 \to a^{11}b$ since $a^3 \cdot a^{11}b = a^{14}b \in H_4$
$a^2 \to a^{12}b$ since $a^2 \cdot a^{12}b = a^{14}b \in H_4$
$a \to a^{13}b$ since $a \cdot a^{13}b = a^{14}b \in H_4$
$a^{15} \to a^{15}b$ since $a^{15} \cdot a^{15}b = a^{14}b \in H_4$
$a^6b \to 1$ since $a^6b \cdot 1 = a^6b \in H_4$
$1 \to a^6b$ since $1 \cdot a^6b = a^6b \in H_4$
$a^{14}b \to a^8$ since $a^{14}b \cdot a^8 = a^6b \in H_4$
$a^8 \to a^{14}b$ since $a^8 \cdot a^{14}b = a^6b \in H_4$
$a^{15}b \to a$ since $a^{15}b \cdot a = a^6b \in H_4$
$a^8b \to a^2$ since $a^8b \cdot a^2 = a^6b \in H_4$
$ab \to a^3$ since $ab \cdot a^3 = a^6b \in H_4$
$a^{10}b \to a^4$ since $a^{10}b \cdot a^4 = a^6b \in H_4$
$a^3b \to a^5$ since $a^3b \cdot a^5 = a^6b \in H_4$
$a^{12}b \to a^6$ since $a^{12}b \cdot a^6 = a^6b \in H_4$
$a^5b \to a^7$ since $a^5b \cdot a^7 = a^6b \in H_4$

$a^3 \to a^9b$ since $a^3 \cdot a^9b = a^{12}b \in H_4$
$a^3 \to a^5$ since $a^3 \cdot a^5 = a^8 \in H_4$
$a^3 \to a^{13}b$ since $a^3 \cdot a^{13}b = b \in H_4$
$a^5 \to a^{11}$ since $a^5 \cdot a^{11} = 1 \in H_4$
$a^5 \to a^7b$ since $a^5 \cdot a^7b = a^{12}b \in H_4$
$a^5 \to a^7$ since $a^5 \cdot a^7 = a^{12} \in H_4$
$a^5 \to a^3$ since $a^5 \cdot a^3 = a^8 \in H_4$
$a^5 \to a^{11}b$ since $a^5 \cdot a^{11}b = b \in H_4$
$a^5 \to a^{15}b$ since $a^5 \cdot a^{15}b = a^4b \in H_4$
$a^5 \to a^{15}$ since $a^5 \cdot a^{15} = a^4 \in H_4$
$a^5 \to a^3b$ since $a^5 \cdot a^3b = a^8b \in H_4$
$a^{10}b \to a^{10}$ since $a^{10}b \cdot a^{10} = b \in H_4$
$a^{10}b \to a^6b$ since $a^{10}b \cdot a^6b = a^4 \in H_4$
$a^{10}b \to a^6$ since $a^{10}b \cdot a^6 = a^4b \in H_4$
$a^{10}b \to a^2$ since $a^{10}b \cdot a^2 = a^8b \in H_4$
$a^{10}b \to a^{14}b$ since $a^{10}b \cdot a^{14}b = a^{12} \in H_4$
$a^{10}b \to a^{14}$ since $a^{10}b \cdot a^{14} = a^{12}b \in H_4$
$a^{10}b \to a^2b$ since $a^{10}b \cdot a^2b = a^8 \in H_4$
$a^{14}b \to a^{10}$ since $a^{14}b \cdot a^{10} = a^4b \in H_4$
$a^{14}b \to a^6b$ since $a^{14}b \cdot a^6b = a^8 \in H_4$
$a^{14}b \to a^6$ since $a^{14}b \cdot a^6 = a^8b \in H_4$
$a^{14}b \to a^2$ since $a^{14}b \cdot a^2 = a^{12}b \in H_4$
$a^{14}b \to a^{10}b$ since $a^{14}b \cdot a^{10}b = a^4 \in H_4$
$a^{14}b \to a^{14}$ since $a^{14}b \cdot a^{14} = b \in H_4$
$a^{14}b \to a^2b$ since $a^{14}b \cdot a^2b = a^{12} \in H_4$
$a^{12}b \to 1$ since $a^{12}b \cdot 1 = a^{12}b \in H_4$
$a^{12}b \to b$ since $a^{12}b \cdot b = a^{12} \in H_4$
$a^{12}b \to a^{12}$ since $a^{12}b \cdot a^{12} = b \in H_4$
$a^{12}b \to a^8$ since $a^{12}b \cdot a^8 = a^4b \in H_4$
$a^{12}b \to a^4b$ since $a^{12}b \cdot a^4b = a^8 \in H_4$
$a^{12}b \to a^8b$ since $a^{12}b \cdot a^8b = a^4 \in H_4$

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$a^7b \to a^9$ since $a^7b \cdot a^9 = a^6b \in H_4$

$b \to a^{10}$ since $b \cdot a^{10} = a^6b \in H_4$

$a^9b \to a^{11}$ since $a^9b \cdot a^{11} = a^6b \in H_4$

$a^2b \to a^{12}$ since $a^2b \cdot a^{12} = a^6b \in H_4$

$a^{11}b \to a^{13}$ since $a^{11}b \cdot a^{13} = a^6b \in H_4$

$a^4b \to a^{14}$ since $a^4b \cdot a^{14} = a^6b \in H_4$

$a^{13}b \to a^{15}$ since $a^{13}b \cdot a^{15} = a^6b \in H_4$

$a^6 \to b$ since $a^6 \cdot b = a^6b \in H_4$

$a^5 \to ab$ since $a^5 \cdot ab = a^6b \in H_4$

$a^4 \to a^2b$ since $a^4 \cdot a^2b = a^6b \in H_4$

$a^3 \to a^3b$ since $a^3 \cdot a^3b = a^6b \in H_4$

$a^2 \to a^4b$ since $a^2 \cdot a^4b = a^6b \in H_4$

$a \to a^5b$ since $a \cdot a^5b = a^6b \in H_4$

$a^{15} \to a^7b$ since $a^{15} \cdot a^7b = a^6b \in H_4$

$a^{14} \to a^8b$ since $a^{14} \cdot a^8b = a^6b \in H_4$

$a^{13} \to a^9b$ since $a^{13} \cdot a^9b = a^6b \in H_4$

$a^{12} \to a^{10}b$ since $a^{12} \cdot a^{10}b = a^6b \in H_4$

$a^{11} \to a^{11}b$ since $a^{11} \cdot a^{11}b = a^6b \in H_4$

$a^{10} \to a^{12}b$ since $a^{10} \cdot a^{12}b = a^6b \in H_4$

$a^9 \to a^{13}b$ since $a^9 \cdot a^{13}b = a^6b \in H_4$

$a^7 \to a^{15}b$ since $a^7 \cdot a^{15}b = a^6b \in H_4$

$a^8 \to 1$ since $a^8 \cdot 1 = a^8 \in H_4$

$1 \to a^8$ since $1 \cdot a^8 = a^8 \in H_4$

$a^7 \to a$ since $a^7 \cdot a = a^8 \in H_4$

$a \to a^7$ since $a \cdot a^7 = a^8 \in H_4$

$a^6 \to a^2$ since $a^6 \cdot a^2 = a^8 \in H_4$

$a^2 \to a^6$ since $a^2 \cdot a^6 = a^8 \in H_4$

$a^5 \to a^3$ since $a^5 \cdot a^3 = a^8 \in H_4$

$a^3 \to a^5$ since $a^3 \cdot a^5 = a^8 \in H_4$

$a^{15} \to a^9$ since $a^{15} \cdot a^9 = a^8 \in H_4$

$a^9 \to a^{15}$ since $a^9 \cdot a^{15} = a^8 \in H_4$

$a^{12}b \to a^4$ since $a^{12}b \cdot a^4 = a^8b \in H_4$

$a^{14} \to a^{10}$ since $a^{14} \cdot a^{10} = a^8 \in H_4$

$a^{14} \to a^6b$ since $a^{14} \cdot a^6b = a^4b \in H_4$

$a^{14} \to a^6$ since $a^{14} \cdot a^6 = a^4 \in H_4$

$a^{14} \to a^2$ since $a^{14} \cdot a^2 = 1 \in H_4$

$a^{14} \to a^{10}b$ since $a^{14} \cdot a^{10}b = a^8b \in H_4$

$a^{14} \to a^{14}b$ since $a^{14} \cdot a^{14}b = a^{12}b \in H_4$

$a^{14} \to a^2b$ since $a^{14} \cdot a^2b = b \in H_4$

$a^{11}b \to a^{11}$ since $a^{11}b \cdot a^{11} = a^8b \in H_4$

$a^{11}b \to a^7b$ since $a^{11}b \cdot a^7b = a^{12} \in H_4$

$a^{11}b \to a^7$ since $a^{11}b \cdot a^7 = a^{12}b \in H_4$

$a^{11}b \to a^3$ since $a^{11}b \cdot a^3 = b \in H_4$

$a^{11}b \to a^{15}b$ since $a^{11}b \cdot a^{15}b = a^4 \in H_4$

$a^{11}b \to a^{15}$ since $a^{11}b \cdot a^{15} = a^4b \in H_4$

$a^{11}b \to a^3b$ since $a^{11}b \cdot a^3b = 1 \in H_4$

$a^{15}b \to a^{11}$ since $a^{15}b \cdot a^{11} = a^{12}b \in H_4$

$a^{15}b \to a^7b$ since $a^{15}b \cdot a^7b = 1 \in H_4$

$a^{15}b \to a^7$ since $a^{15}b \cdot a^7 = b \in H_4$

$a^{15}b \to a^3$ since $a^{15}b \cdot a^3 = a^4b \in H_4$

$a^{15}b \to a^{11}b$ since $a^{15}b \cdot a^{11}b = a^{12} \in H_4$

$a^{15}b \to a^{15}$ since $a^{15}b \cdot a^{15} = a^8b \in H_4$

$a^{15}b \to a^3b$ since $a^{15}b \cdot a^3b = a^4 \in H_4$

$a^{13}b \to a$ since $a^{13}b \cdot a = a^4b \in H_4$

$a^{13}b \to ab$ since $a^{13}b \cdot ab = a^4 \in H_4$

$a^{13}b \to a^{13}$ since $a^{13}b \cdot a^{13} = a^8b \in H_4$

$a^{13}b \to a^9$ since $a^{13}b \cdot a^9 = a^{12}b \in H_4$

$a^{13}b \to a^5b$ since $a^{13}b \cdot a^5b = 1 \in H_4$

$a^{13}b \to a^9b$ since $a^{13}b \cdot a^9b = a^{12} \in H_4$

$a^{13}b \to a^5$ since $a^{13}b \cdot a^5 = b \in H_4$

$a^{15} \to a$ since $a^{15} \cdot a = 1 \in H_4$

$a^{15} \to ab$ since $a^{15} \cdot ab = b \in H_4$

$a^{14} \to a^{10}$ since $a^{14} \cdot a^{10} = a^8 \in H_4$

$a^{10} \to a^{14}$ since $a^{10} \cdot a^{14} = a^8 \in H_4$

$a^{13} \to a^{11}$ since $a^{13} \cdot a^{11} = a^8 \in H_4$

$a^{11} \to a^{13}$ since $a^{11} \cdot a^{13} = a^8 \in H_4$

$a^8b \to b$ since $a^8b \cdot b = a^8 \in H_4$

$b \to a^8b$ since $b \cdot a^8b = a^8 \in H_4$

$a^{10}b \to a^2b$ since $a^{10}b \cdot a^2b = a^8 \in H_4$

$a^2b \to a^{10}b$ since $a^2b \cdot a^{10}b = a^8 \in H_4$

$a^{12}b \to a^4b$ since $a^{12}b \cdot a^4b = a^8 \in H_4$

$a^4b \to a^{12}b$ since $a^4b \cdot a^{12}b = a^8 \in H_4$

$a^{14}b \to a^6b$ since $a^{14}b \cdot a^6b = a^8 \in H_4$

$a^6b \to a^{14}b$ since $a^6b \cdot a^{14}b = a^8 \in H_4$

$a^{12} \to 1$ since $a^{12} \cdot 1 = a^{12} \in H_4$

$1 \to a^{12}$ since $1 \cdot a^{12} = a^{12} \in H_4$

$a^4 \to a^8$ since $a^4 \cdot a^8 = a^{12} \in H_4$

$a^8 \to a^4$ since $a^8 \cdot a^4 = a^{12} \in H_4$

$a^{11} \to a$ since $a^{11} \cdot a = a^{12} \in H_4$

$a \to a^{11}$ since $a \cdot a^{11} = a^{12} \in H_4$

$a^{10} \to a^2$ since $a^{10} \cdot a^2 = a^{12} \in H_4$

$a^2 \to a^{10}$ since $a^2 \cdot a^{10} = a^{12} \in H_4$

$a^{15} \to a^{13}$ since $a^{15} \cdot a^{13} = a^{12} \in H_4$

$a^{15} \to a^9$ since $a^{15} \cdot a^9 = a^8 \in H_4$

$a^{15} \to a^5b$ since $a^{15} \cdot a^5b = a^4b \in H_4$

$a^{15} \to a^9b$ since $a^{15} \cdot a^9b = a^8b \in H_4$

$a^{15} \to a^5$ since $a^{15} \cdot a^5 = a^4 \in H_4$

$a^{15} \to a^{13}b$ since $a^{15} \cdot a^{13}b = a^{12}b \in H_4$

$a^2b \to a^{10}$ since $a^2b \cdot a^{10} = a^8b \in H_4$

$a^2b \to a^6b$ since $a^2b \cdot a^6b = a^{12} \in H_4$

$a^2b \to a^6$ since $a^2b \cdot a^6 = a^{12}b \in H_4$

$a^2b \to a^2$ since $a^2b \cdot a^2 = b \in H_4$

$a^2b \to a^{10}b$ since $a^2b \cdot a^{10}b = a^8 \in H_4$

$a^2b \to a^{14}b$ since $a^2b \cdot a^{14}b = a^4 \in H_4$

$a^2b \to a^{14}$ since $a^2b \cdot a^{14} = a^4b \in H_4$

$a^3b \to a^{11}$ since $a^3b \cdot a^{11} = b \in H_4$

$a^3b \to a^7b$ since $a^3b \cdot a^7b = a^4 \in H_4$

$a^3b \to a^7$ since $a^3b \cdot a^7 = a^4b \in H_4$

$a^3b \to a^3$ since $a^3b \cdot a^3 = a^8b \in H_4$

$a^3b \to a^{11}b$ since $a^3b \cdot a^{11}b = 1 \in H_4$

$a^3b \to a^{15}b$ since $a^3b \cdot a^{15}b = a^{12} \in H_4$

$a^3b \to a^{15}$ since $a^3b \cdot a^{15} = a^{12}b \in H_4$

Therefore, $\Gamma_{H_4}^{NN}(QD_{32})$ is the graph shown in Figure 10.

Based on Propositions 1 to 4, the non-normal cyclic subgroup graph of the quasidihedral groups of order 32 is observed to be the union of some connected graphs.

Conclusion

This study focuses on the determination of the non-normal cyclic subgroup graphs of the quasidihedral group, specifically QD_{32}, where $n = 5$. The results show that the non-normal cyclic subgroup graphs of the quasidihedral group of order 32 can be classified into four different types, each of which may be viewed as a union of several connected graphs. Future work may extend this approach to quasidihedral groups of other orders, and subsequent studies may aim to establish a general form for the non-normal cyclic subgroup graphs of quasidihedral groups.

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Appendix 1

```
gap> F:=FreeGroup("a","b");
<free group on the generators [ a, b ]>
gap> a:=F.1;b:=F.2;
a
b
gap> G:=F/[a^8,b^2,b*a*b*a^5];
<fp group on the generators [ a, b ]>
gap> a:=G.1;b:=G.2;
a
b
gap> Order(G);
16
gap> e:=Elements(G);
[ <identity ...>, <[ [ 1, -2, 1, -1, 1, -1, 1, -1 ], [ 2, 3 ] ]|a^-15>, b,
  <[ [ 1, -2, 1, -1, 1, -1, 1, -1 ], [ 2, 2 ] ]|a^-10>, <[ [ 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 4 ] ]|a^20>, a^-15*b,
  <[ [ 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 3 ] ]|a^15>, <[ [ 1, 1, 1, 1, 1, 1, 1, 2 ] ]|a^5>, a^10*b, a^20*b,
  <[ [ 1, 1, 1, 1, 1, 1, 1, 2 ], [ 2, 2 ] ]|a^10>, a^-5*b, a^5*b, <[ [ 1, -2, 1, -1, 1, -1, 1, -1 ] ]|a^-5>, a^-10*b,
  a^15*b ]
gap> AllSubgroups(G);
[ Group([  ]), Group([ a^20 ]), Group([ b ]), Group([ a^-1*b*a ]), Group([ a^-2*b*a^2 ]), Group([ b^-1*a^-1*b*a*b ]),
  Group(<fp, no generators known>), Group([ b, a*b*a ]), Group([ a^2*b^-1, a*b*a^-1 ]), Group([ b*a^-1 ]),
  Group([ b*a ]), Group(<fp, no generators known>), Group(<fp, no generators known>), Group(<fp, no generators known>)
    , Group([ a, b ]) ]
gap> N:=NormalSubgroups(G);
[ Group(<fp, no generators known>), Group(<fp, no generators known>), Group(<fp, no generators known>),
  Group(<fp, no generators known>), Group(<fp, no generators known>), Group(<fp, no generators known>),
  Group(<fp, no generators known>) ]
gap> Elements(N);
[ Group([  ]), Group([ a, b ]), Group([ a ]), Group([ a^-2, b ]), Group([ a^-2, b*a^-1 ]), Group([ a^-2 ]),
  Group([ a^-4 ]) ]
gap> H1:=Subgroup(G,[b]);
Group([ b ])
gap> Elements(H1);
[ <identity ...>, <[ [ 1, 1 ] ]|b> ]
gap> for i in e do
> for j in e do;
> if i<>j then
> c:=i*j;
> d:=c in H1;
> if (d=true) then
> Print("{",i,",",j,",",c,",",d,"},");
> fi;
> fi;
> od;
> od;
```

```
{<identity ...>,b,b,true},{a^-15,a^15,<identity ...>,true},{a^-15,a^15*b,b,true},{b,<identity ...>,b,true},
b*a^5,true},{a^-15*b,a^5*b,a^-15*b*a^5*b,true},{a^15,a^-15,<identity ...>,true},{a^15,a^-15*b,b,true},{a^5,
0,true},{a^10,a^-10,<identity ...>,true},{a^10,a^-10*b,b,true},{a^-5*b,a^15,a^-5*b*a^15,true},{a^-5*b,a^15*
dentity ...>,true},{a^-5,a^5*b,b,true},{a^-10*b,a^-10,a^-10*b*a^-10,true},{a^15*b,a^-5*b,a^15*b*a^-5*b,true
```

```
,{a^-10,a^10*b,b,true},{a^-10,a^10,<identity ...>,true},{a^20,a^20*b,a^40*b,true},{a^-15*b,a^5,a^-15*\
,a^-5*b,b,true},{a^5,a^-5,<identity ...>,true},{a^10*b,a^10,a^10*b*a^10,true},{a^20*b,a^20,a^20*b*a^2\
*b,a^-5*b*a^15*b,true},{a^5*b,a^-15,a^5*b*a^-15,true},{a^5*b,a^-15*b,a^5*b*a^-15*b,true},{a^-5,a^5,<i\
e},{a^15*b,a^-5,a^15*b*a^-5,true},
```

Appendix 2

```
> NH1 := Graph([1, a, a2, a3, a4, a5, a6, a7, b, ab, a2b, a3b, a4b, a5b, a6b, a7b], {[b, 1], [1, b], [a4b, a4], [a4, a4b], [a7, a],
    [a, a7], [a6, a2], [a2, a6], [a5, a3], [a3, a5], [a5b, ab], [ab, a5b], [a7b, a3b], [a3b, a7b], [a5b, a], [a2b, a2], [a7b, a3],
    [ab, a5], [a6b, a6], [a3b, a7], [a7, ab], [a6, a2b], [a5, a3b], [a3, a5b], [a2, a6b], [a, a7b]})
                NH1 := Graph 1: a directed unweighted graph with 16 vertices and 26 arc(s)
> DrawGraph(NH1, size = [500, 500], HighlightEdges(NH1, Edges(NH1), black))
```