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(IJIREV)**www.gaexcellence.com/ijirev**A NEW DUAL AND SYSTEM FOR INTERVAL TYPE-2
FUZZY PENTAGONAL POLYNOMIALS**Nurhakimah Ab Rahman^{1*}, Nuraini Rahim², Lazim Abdullah²¹STEM Foundation Centre, Universiti Malaysia Terengganu, Malaysia nurhakimah@umt.edu.my <https://orcid.org/0000-0002-0362-2505>²Faculty of Computer Science and Mathematics, Universiti Malaysia Terengganu, Malaysia snaini276@gmail.com <https://orcid.org/0000-0003-4421-795X>²Faculty of Computer Science and Mathematics, Universiti Malaysia Terengganu, Malaysia lazim_m@umt.edu.my <https://orcid.org/0000-0002-6646-4751>

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DOI: 10.35631/IJIREV.826024**Abstract:**

Duality plays an important role in representing incomplete information, uncertainty, and imprecision in complex event modelling. In addition, polynomial systems provide a useful mathematical framework for representing uncertainty in real-world problems. This paper proposes a new interval type-2 dual fuzzy pentagonal polynomial and a system of interval type-2 fuzzy pentagonal polynomials. The proposed formulations integrate the characteristics of interval type-2 fuzzy sets with pentagonal fuzzy numbers in dual polynomial equations and polynomial systems. An iterative Horner–Muller method is then applied to the proposed interval type-2 fuzzy pentagonal polynomial formulations to obtain approximate solutions. The computational procedures for both the dual polynomial equation and the polynomial system are demonstrated through numerical examples. The results illustrate the implementation of the proposed approach for solving interval type-2 dual fuzzy pentagonal polynomial equations and systems of interval type-2 fuzzy pentagonal polynomial equations.

Keyword:

Dual-Systems; Fuzzy Polynomials; Horner-Muller's Method; Interval Type-2 Fuzzy Numbers; Pentagonal Fuzzy Number; Polynomials



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Introduction

The natural phenomenon is uncertain and difficult to define with a crisp value. In order to handle this uncertainty of information in real-life problems (Zadeh, 1965) introduced the fuzzy sets theory which considers membership degrees that are defined by $[0,1]$. The fundamental concept of fuzzy sets and numbers was developed by Chang and Zadeh (1972), where a fuzzy number is a tool for representing an imprecise notion. The fuzzy sets have been extended to interval-valued fuzzy sets by (Sambuc, 1975), intuitionistic fuzzy sets by (Atanassov, 1986) and interval type-2 fuzzy sets by (Qilian Liang & Mendel, 2000).

Qilian Liang and Mendel (2000) deal with problems of uncertainty, imprecision and vagueness. Zadeh (1975) first explicitly introduced the notion of interval type-2 fuzzy sets, which assert that each element's membership degree is determined by another fuzzy set defined over the referential $[0,1]$. The definition of IT2 FS was presented by Liang and Mendel (2000) as a specific example of the concept of type-2 fuzzy sets being mathematically defined when the secondary membership degree is equal to one. Sola *et al.* (2015) provided a more comprehensive overview of the relationship between interval-valued fuzzy sets and IT2 FS based on the generalization of interval-valued fuzzy sets. It is generally acknowledged that interval type-2 fuzzy numbers can be used in a wider range of situations and make mathematical calculations easier (Qilian Liang & Mendel, 2000).

Prior studies have frequently defined IT2 FS with fuzzy numbers, especially triangular, trapezoidal, and pentagonal fuzzy numbers. There are some fuzzy numbers, including triangular fuzzy numbers (Vidhya & Ganesan, 2022), trapezoidal fuzzy numbers (Chang & Zadeh, 1972; Bansal, 2011), pentagonal fuzzy numbers (Gani & Assarudeen, 2012), hexagonal fuzzy numbers, octagonal fuzzy numbers, pyramid fuzzy numbers and diamond fuzzy numbers (Pathinathan & Ponnivalavan, 2015) have been introduced with their membership function. There are several real-world issues that involve more than four tuples (Ponnivalavan & Pathinathan, 2014). To address these issues, Pathinathan *et al.* (2015) introduced pentagonal fuzzy sets along with certain operational properties. The concepts of pentagonal fuzzy numbers and set theoretic operations were also presented by Kumar and Pathinathan (2015). They proposed the concept of fuzzy centers in pentagonal fuzzy numbers. Kaur (2018) defines pentagonal fuzzy numbers with the other fuzzy numbers. Pentagonal fuzzy numbers were presented by Vidhya and Ganesan (2022) to address the fuzzy transportation problem without requiring an initial feasible solution.

The study of pentagonal fuzzy numbers has been integrated with interval-valued fuzzy numbers. Chakraborty *et al.* (2019) studies the different measures of interval-valued pentagonal fuzzy numbers connected to various membership functions by considering the significant exposure of multifarious interval-valued fuzzy numbers in neoteric studies. The interval-valued pentagonal fuzzy number has been introduced by Ajay (2018) and applied in the literature. Ajay and Pathinathan (2019) introduce interval type-2 pentagonal fuzzy numbers and analyze three types of regular interval type-2 pentagonal fuzzy numbers from the generalization of pentagonal fuzzy numbers and interval-valued pentagonal fuzzy numbers. Some researchers have shown an interest in fuzzy polynomials using trapezoidal and triangular fuzzy numbers. However, the combination of interval type-2 pentagonal fuzzy numbers with polynomials has not been applied by any reliable information.

Mathematical expressions known as polynomials are composed of a sum of terms, each of which includes a variable or a variable raised to a power and multiplied by a coefficient. Fuzzy numbers become the coefficients of fuzzy polynomials, which nonetheless maintain all of the other properties of a polynomial. At first, Rouhparvar (2007) and Noor'ani *et al.* (2011) introduced fuzzy polynomials. The interval type-2 fuzzy polynomial equation was presented by Nurhakimah and Abdullah (2014) and solved it using the ranking method. In order to solve interval type-2 fuzzy polynomials using the ranking method, the concepts of value, ambiguity, fuzziness, and vagueness are applied (Nurhakimah *et al.*, 2015). Fuzzy polynomials can be divided into three categories: fuzzy polynomial systems, dual fuzzy polynomial equations, and fuzzy polynomial equations.

Today, duality is an important element in the display of incomplete information, uncertainties and imprecisions in complex event modelling. The concept of dual hesitant fuzzy sets was discussed by Zhu *et al.* (2012) and presented by Chen and Huang (2017) with membership and non-membership grades. Arockiaraj and Revathy (2015) solved numerically some problems of mensuration based on operations using dual trapezoidal fuzzy numbers and calculated the approximate values. The applicability of the multi-criteria decision-making technique using generalized dual hesitant hexagonal fuzzy numbers studied by Momena *et al.* (2023) for the diagnosis of the patient. The concept of a ranking method is proposed by Noor'ani *et al.* (2011) to find the real roots of a dual fuzzy polynomial equation by ranking fuzzy numbers. Razvarz *et al.* (2018) obtained the approximate solution of fully fuzzy dual polynomials using a fuzzy neural network. Nurhakimah and Abdullah (2014) proposed an interval type-2 dual fuzzy polynomial equation and applied the ranking method to find real roots.

Systems of polynomials are widely used in many fields, including the social sciences, engineering, and both pure and applied mathematics (Abbasbandy & Amirfakhrian, 2006). Tu and Peng (2010) defined fuzzy polynomial equations and systems of fuzzy polynomial equations. To solve the system of fuzzy polynomials that is converted to a system of crisp polynomials, Noor'ani *et al.* (2010) presented a ranking fuzzy number approach. A fuzzy system method is presented by Aubry *et al.* (2020) for finding real solutions of polynomial systems. Li *et al.* (2015a) studied the issue of filter design for interval type-2 fuzzy systems with D stability constraints using a novel performance index. For interval type-2 fuzzy systems with mismatched membership functions, Li (2015b) discussed the issues of state and output feedback control. The issues of stabilization for interval type-2 polynomial fuzzy systems are explored by Li *et al.* (2017) with time-varying delay and parameter uncertainty.

Preliminaries

This section introduces the definitions of related research on interval type-2 fuzzy sets, interval type-2 pentagonal fuzzy number, pentagonal fuzzy number, interval type-2 pentagonal fuzzy number and fuzzy polynomials.

Definition of Interval Type-2 Fuzzy Sets (Mendel *et al.*, 2006)

A type-2 fuzzy sets, A_{T2FS} is characterized by a type-2 membership function, $\mu_{A_{T2FS}}(x, u)$ where $x \in X$ and $u \in J_x \subseteq [0, 1]$. It is defined as:

$$A_{T2FS} = \left\{ ((x, u), \mu_{A_{T2FS}}(x, u)) / \forall x \in X, \forall u \in J_x \subseteq [0, 1] \right\}$$

where J_x is the primary membership function in $(0,1)$ and u is the primary membership values in $0 \leq \mu_{A_{IT2FS}}(x,u) \leq 1$. It can also be expressed as follows:

$$A_{IT2FS} = \int_{x \in X} \int_{u \in J_x} \mu_{A_{IT2FS}}(x,u) / (x,u); J_x \subseteq [0,1]$$

where $\int \int$ denotes union over all admissible x and u . An interval type-2 fuzzy set A_{IT2FS} is a type-2 fuzzy set when all secondary membership function is unity defined as:

$$A_{IT2FS} = \int_{x \in X} \int_{u \in J_x} 1 / (x,u); J_x \subseteq [0,1]$$

Definition of Interval Type-2 Fuzzy Number (Mendel *et al.*, 2006)

An interval type-2 fuzzy set defined on real line R is called interval type-2 fuzzy number A_{IT2FN} . It is defined as:

$$A_{IT2FN} = [A^U, A^L]$$

where A^U and A^L are upper and lower membership functions respectively such that $A^U \subseteq A^L$.

Definition of Pentagonal Fuzzy Number (Momena *et al.*, 2023)

A pentagonal fuzzy number, A_{PFN} is defined as:

$$A_{PFN} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5)$$

where the condition is satisfied as follows:

$\mu_{A_{PFN}}(x)$ is a continuous function in the interval $[0,1]$

$\mu_{A_{PFN}}(x)$ is strictly increasing and continuous function on $[\tilde{a}_1, \tilde{a}_2]$ and $[\tilde{a}_2, \tilde{a}_3]$

$\mu_{A_{PFN}}(x)$ is strictly decreasing and continuous function on $[\tilde{a}_3, \tilde{a}_4]$ and $[\tilde{a}_4, \tilde{a}_5]$

Definition of Interval Type-2 Pentagonal Fuzzy Number (Ponnivalavan & Pathinathan, 2014)

An IT2PFN is an interval type-2 fuzzy set of R . It is defined as

$$A_{IT2PFN} = [\tilde{A}^U, \tilde{A}^L]$$

where $\tilde{A}^U = (\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5)$ and $\tilde{A}^L = (\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5)$ are upper and lower IT2PFN respectively such that $\tilde{A}^U \subseteq \tilde{A}^L$.

According to this definition, it can be concluded that the parametric form for interval type-2 pentagonal fuzzy numbers can be obtained as below:

$$\begin{aligned} \mu(x) &= \underline{a}_1 - \underline{a}_2 + \underline{a}_3 \\ \bar{\mu}(x) &= \bar{a}_3 - \bar{a}_4 + \bar{a}_5 \end{aligned}$$

Definition of Fuzzy Polynomial (Zhu *et al.*, 2012)

Polynomial can be written as $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$. Then, a fuzzy polynomial equation can be defined as follows:

$$A_1 x + A_2 x^2 + \dots + A_n x^n = A_0$$

where $x \in \Re$, the coefficient $A_1, A_2, \dots, A_n$ and A_0 are fuzzy numbers.

Computational Procedures

This section provides the proposed computational procedures to solve the IT2 DFPP and IT2 FPPS using Horner-Muller's method in this study. Firstly, the new dual and system for IT2 FPP are proposed with defines the development of IT2 DFPP and IT2 FPPS. Then, Horner-Muller's method is applied where eventually the approximate solution can be obtained. Figure 1 presents the general framework of computational procedures in this study.

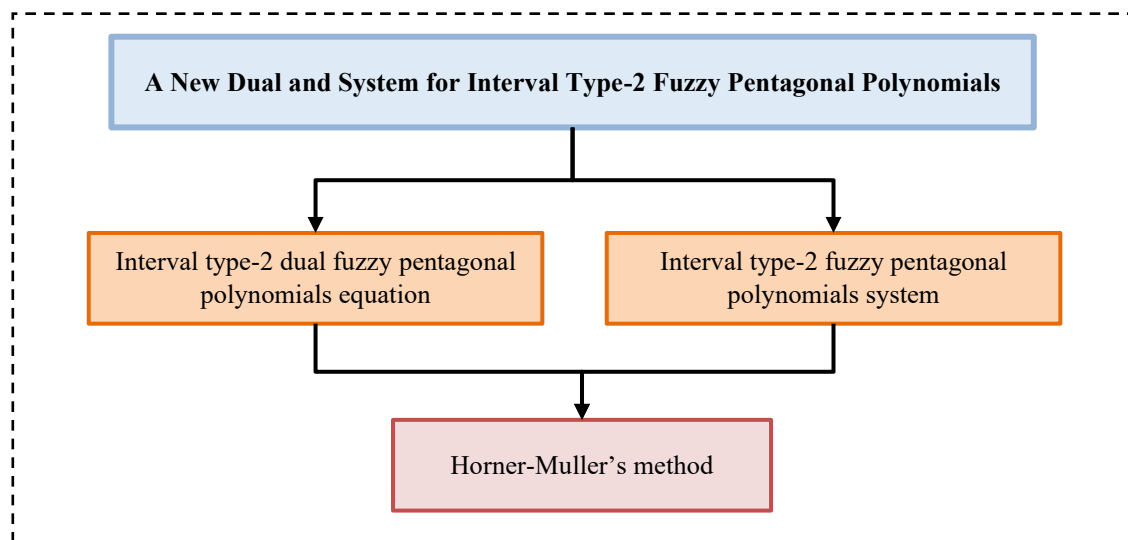


Figure 1: General Framework of Computational Procedures

Iterative methods such as Horner's and Muller's methods have been created to model and address real-world problems. The combination of Horner's and Muller's methods is called Horner-Muller's method developed by Nurhakimah *et al.* (2024) in order to more effectively find the approximate solution for solving interval type-2 fuzzy polynomials. The combination of these two iterative methods has not been used to solve IT2 DFPP and IT2 FPPS. Therefore, we applied Horner-Muller's method to solve the IT2 DFPP and IT2 FPPS. The detailed computational procedure of this study is presented in Figure 2.

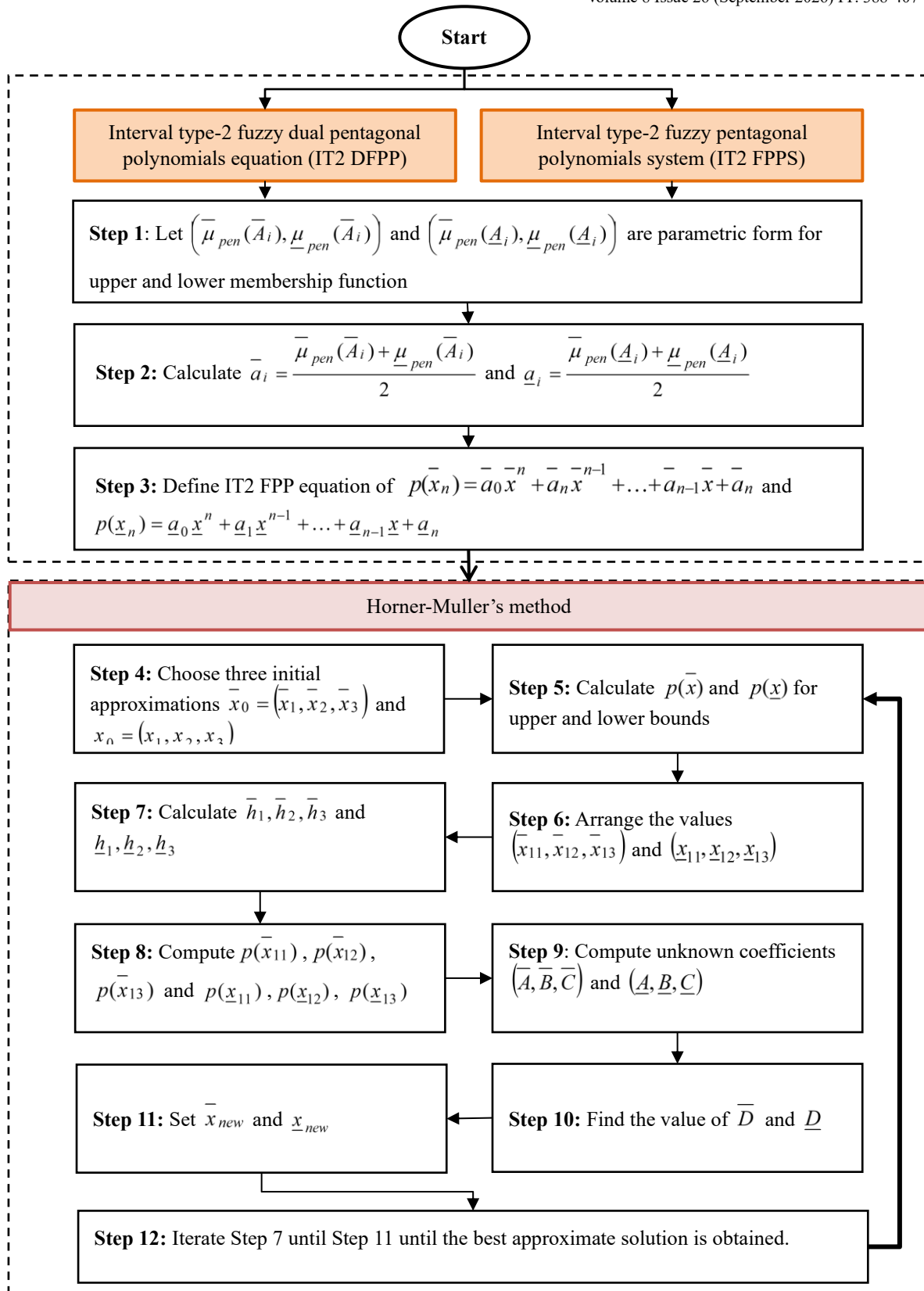


Figure 2: The Details of Computational Procedures

Interval Type-2 Dual Fuzzy Pentagonal Polynomial Equations

This section defines the dual of interval type-2 fuzzy pentagonal polynomials that are generated from the definition of interval type-2 pentagonal fuzzy numbers, definition of fuzzy polynomials and dual polynomial equation. We proposed IT2 DFPP and perform solution for this proposed using Horner-Muller's method to find the approximate solution of IT2 DFPP equation. IT2 DFPP consists of two nonlinear equations situated on the left and right side of the equation, respectively.

Definition of Dual Polynomial Equation (Noor'ani, 2011)

The polynomial equation

$$A_1'x + A_2'x^2 + \dots + A_n'x^n = A_1''x + A_2''x^2 + \dots + A_n''x^n + A_0$$

where $x \in R$. The coefficient $A_1', A_2', A_n', A_1'', A_2'', A_n''$ and A_0 are interval type-2 fuzzy pentagonal number.

Hence, the IT2 DFPP is defined as follows:

$$\begin{aligned} (A_1^U, A_1^L)'x + (A_2^U, A_2^L)'x^2 + \dots + (A_n^U, A_n^L)'x^n &= (A_1^U, A_1^L)''x + (A_2^U, A_2^L)''x^2 + \dots + (A_n^U, A_n^L)''x^n + (A_0^U, A_0^L) \\ \text{where } (A^U)' &= (\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5)', (A^L)' = (\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5)', (A^U)'' = (\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5)'' \text{ and} \\ (A^L)'' &= (\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5)'' \end{aligned}$$

Therefore, it also be written as follows:

$$\begin{aligned} &\left\{ (\bar{a}_{11}, \bar{a}_{21}, \bar{a}_{31}, \bar{a}_{41}, \bar{a}_{51})', (\underline{a}_{11}, \underline{a}_{21}, \underline{a}_{31}, \underline{a}_{41}, \underline{a}_{51})' \right\} x + \left\{ (\bar{a}_{12}, \bar{a}_{22}, \bar{a}_{32}, \bar{a}_{42}, \bar{a}_{52})', (\underline{a}_{12}, \underline{a}_{22}, \underline{a}_{32}, \underline{a}_{42}, \underline{a}_{52})' \right\} x^2 + \dots + \\ &\left\{ (\bar{a}_{1n}, \bar{a}_{2n}, \bar{a}_{3n}, \bar{a}_{4n}, \bar{a}_{5n})', (\underline{a}_{1n}, \underline{a}_{2n}, \underline{a}_{3n}, \underline{a}_{4n}, \underline{a}_{5n})' \right\} x^n = \left\{ (\bar{a}_{11}, \bar{a}_{21}, \bar{a}_{31}, \bar{a}_{41}, \bar{a}_{51})'', (\underline{a}_{11}, \underline{a}_{21}, \underline{a}_{31}, \underline{a}_{41}, \underline{a}_{51})'' \right\} x + \\ &\left\{ (\bar{a}_{12}, \bar{a}_{22}, \bar{a}_{32}, \bar{a}_{42}, \bar{a}_{52})'', (\underline{a}_{12}, \underline{a}_{22}, \underline{a}_{32}, \underline{a}_{42}, \underline{a}_{52})'' \right\} \left\{ (\bar{a}_{1n}, \bar{a}_{2n}, \bar{a}_{3n}, \bar{a}_{4n}, \bar{a}_{5n})'', (\underline{a}_{1n}, \underline{a}_{2n}, \underline{a}_{3n}, \underline{a}_{4n}, \underline{a}_{5n})'' \right\} x^2 + \dots + \\ &\left\{ (\bar{a}_0, \bar{a}_0, \bar{a}_0, \bar{a}_0, \bar{a}_0), (\underline{a}_0, \underline{a}_0, \underline{a}_0, \underline{a}_0, \underline{a}_0) \right\} \end{aligned}$$

The IT2 DFPP equation should have two nonlinear equations where the first and second equations are situated on the left and right side of the equation. We illustrate numerical examples for considering interval type-2 dual fuzzy pentagonal polynomial equations using the Horner-Muller method based on Figure 2 to find its approximate solution.

Example 1: Let we consider the interval type-2 dual fuzzy pentagonal polynomial equations as follows:

$$\begin{aligned} & \left\{ (0.1, 0.2, 0.3, 0.4, 0.5)' , (0.2, 0.3, 0.4, 0.5, 0.6)' \right\} x + \left\{ (0.2, 0.3, 0.4, 0.5, 0.6)' , (0.25, 0.35, 0.45, 0.55, 0.65)' \right\} x^2 + \\ & \left\{ (0.3, 0.4, 0.5, 0.6, 0.7)' , (0.35, 0.4, 0.45, 0.5, 0.55)' \right\} x^3 + \left\{ (0.4, 0.5, 0.6, 0.7, 0.8)' , (0.6, 0.7, 0.8, 0.9, 1.0)' \right\} x^4 = \\ & \left\{ (0.05, 0.1, 0.15, 0.2, 0.25)'' , (0.25, 0.35, 0.45, 0.55, 0.65)'' \right\} x + \left\{ (0.15, 0.2, 0.25, 0.3, 0.35)'' , (0.2, 0.25, 0.3, 0.35, 0.4)'' \right\} x^2 + \\ & \left\{ (0.25, 0.3, 0.35, 0.4, 0.45)'' , (0.3, 0.35, 0.4, 0.45, 0.5)'' \right\} x^3 + \left\{ (0.35, 0.4, 0.45, 0.5, 0.55)'' , (0.4, 0.45, 0.5, 0.55, 0.6)'' \right\} x^4 + \\ & \left\{ (0.2, 0.4, 0.6, 0.8, 1) , (0.25, 0.35, 0.45, 0.55, 0.65) \right\} \end{aligned}$$

Step 1: The parametric form for the upper membership functions is given as:

$$\begin{aligned} \left[\bar{A}_1 \right]_l^\alpha &= 0.1 - 0.2 + (0.3)(0.5) = 0.05 & \left[\bar{A}_1 \right]_U^\alpha &= 0.3 - 0.4 + (0.5)(0.5) = 0.15 \\ \left[\bar{A}_2 \right]_l^\alpha &= 0.2 - 0.3 + (0.4)(0.5) = 0.1 & \left[\bar{A}_2 \right]_U^\alpha &= 0.4 - 0.5 + (0.6)(0.5) = 0.2 \\ \left[\bar{A}_3 \right]_l^\alpha &= 0.3 - 0.4 + (0.5)(0.5) = 0.15 & \left[\bar{A}_3 \right]_U^\alpha &= 0.5 - 0.6 + (0.7)(0.5) = 0.25 \\ \left[\bar{A}_4 \right]_l^\alpha &= 0.4 - 0.5 + (0.6)(0.5) = 0.2 & \left[\bar{A}_4 \right]_U^\alpha &= 0.6 - 0.7 + (0.8)(0.5) = 0.3 \\ \\ \left[\bar{A}_1 \right]_l^\alpha &= 0.05 - 0.1 + (0.15)(0.5) = 0.025 & \left[\bar{A}_1 \right]_U^\alpha &= 0.15 - 0.2 + (0.25)(0.5) = 0.075 \\ \left[\bar{A}_2 \right]_l^\alpha &= 0.15 - 0.2 + (0.25)(0.5) = 0.075 & \left[\bar{A}_2 \right]_U^\alpha &= 0.25 - 0.3 + (0.35)(0.5) = 0.125 \\ \left[\bar{A}_3 \right]_l^\alpha &= 0.25 - 0.3 + (0.35)(0.5) = 0.125 & \left[\bar{A}_3 \right]_U^\alpha &= 0.35 - 0.4 + (0.45)(0.5) = 0.175 \\ \left[\bar{A}_4 \right]_l^\alpha &= 0.35 - 0.4 + (0.45)(0.5) = 0.175 & \left[\bar{A}_4 \right]_U^\alpha &= 0.45 - 0.5 + (0.55)(0.5) = 0.225 \\ \\ \left[\bar{A}_0 \right]_l^\alpha &= 0.2 - 0.4 + (0.6)(0.5) = 0.1 & \left[\bar{A}_0 \right]_U^\alpha &= 0.6 - 0.8 + (1)(0.5) = 0.3 \end{aligned}$$

and the parametric form for the lower membership functions is given as:

$$\begin{aligned} \left[\underline{A}_1 \right]_l^\alpha &= 0.2 - 0.3 + (0.4)(0.5) = 0.1 & \left[\underline{A}_1 \right]_U^\alpha &= 0.3 - 0.4 + (0.5)(0.5) = 0.2 \\ \left[\underline{A}_2 \right]_l^\alpha &= 0.25 - 0.35 + (0.45)(0.5) = 0.125 & \left[\underline{A}_2 \right]_U^\alpha &= 0.4 - 0.5 + (0.6)(0.5) = 0.225 \\ \left[\underline{A}_3 \right]_l^\alpha &= 0.35 - 0.4 + (0.45)(0.5) = 0.175 & \left[\underline{A}_3 \right]_U^\alpha &= 0.5 - 0.6 + (0.7)(0.5) = 0.225 \\ \left[\underline{A}_4 \right]_l^\alpha &= 0.6 - 0.7 + (0.8)(0.5) = 0.3 & \left[\underline{A}_4 \right]_U^\alpha &= 0.4 - 0.5 + (0.6)(0.5) = 0.4 \\ \\ \left[\underline{A}_1 \right]_l^\alpha &= 0.25 - 0.35 + (0.45)(0.5) = 0.125 & \left[\underline{A}_1 \right]_U^\alpha &= 0.15 - 0.2 + (0.25)(0.5) = 0.225 \\ \left[\underline{A}_2 \right]_l^\alpha &= 0.2 - 0.25 + (0.3)(0.5) = 0.1 & \left[\underline{A}_2 \right]_U^\alpha &= 0.25 - 0.3 + (0.35)(0.5) = 0.15 \\ \left[\underline{A}_3 \right]_l^\alpha &= 0.3 - 0.35 + (0.4)(0.5) = 0.15 & \left[\underline{A}_3 \right]_U^\alpha &= 0.35 - 0.4 + (0.45)(0.5) = 0.2 \\ \left[\underline{A}_4 \right]_l^\alpha &= 0.4 - 0.45 + (0.5)(0.5) = 0.2 & \left[\underline{A}_4 \right]_U^\alpha &= 0.45 - 0.5 + (0.55)(0.5) = 0.25 \\ \\ \left[\underline{A}_0 \right]_l^\alpha &= 0.25 - 0.35 + (0.45)(0.5) = 0.125 & \left[\underline{A}_0 \right]_U^\alpha &= 0.6 - 0.8 + (1)(0.5) = 0.225 \end{aligned}$$

Step 2: Divide the parametric forms of the upper membership functions and the lower membership functions by 2.

$$\begin{aligned} \bar{a}_1 &= \frac{0.05 + 0.15 + 0.025 + 0.075}{2} = 0.15, & \bar{a}_2 &= \frac{0.1 + 0.2 + 0.075 + 0.125}{2} = 0.25, \\ \bar{a}_3 &= \frac{0.15 + 0.25 + 0.125 + 0.175}{2} = 0.35, & \bar{a}_4 &= \frac{0.2 + 0.3 + 0.175 + 0.225}{2} = 0.45, \\ \bar{a}_0 &= \frac{0.2 + 0.3 + 0.1 + 0.3 +}{2} = 0.2, \end{aligned}$$

and $\underline{a}_1 = 0.325$, $\underline{a}_2 = 0.3$, $\underline{a}_3 = 0.375$, $\underline{a}_4 = 0.575$, $\underline{a}_0 = 0.175$

Step 3: Therefore, the IT2 FPP is defined as follows:

$$p(\bar{x}) = 0.45x^4 + 0.35x^3 + 0.25x^2 + 0.15x - 0.2 \quad \text{and} \quad p(\underline{x}) = 0.575x^4 + 0.375x^3 + 0.3x^2 + 0.325x - 0.175$$

Step 4: We choose three initial approximations as below:

$$\bar{x}_1 = \underline{x}_1 = 0.15, \quad \bar{x}_2 = \underline{x}_2 = 0.55, \quad \bar{x}_3 = \underline{x}_3 = 0.85$$

Step 5: For the upper membership function, we substitute the initial values $\bar{x}_{11}$, $\bar{x}_{12}$ and $\bar{x}_{13}$ consecutively into the following arrangements:

$\bar{x}_1 = 0.15$	0.45	0.35	0.25	0.15	0.20
		0.06750	0.06263	0.04689	0.02953
$\bar{x}_1 = 0.15$	0.45	0.41750	0.31263	0.19689	-0.17047
		0.06750	0.07275	0.05781	
	0.45	0.48500	0.38538	0.25470	

$$p(\bar{x}_1) = -0.17047 \text{ and } p'(\bar{x}_1) = 0.25470, \text{ hence } \bar{x}_{11} = 0.15 - \left(\frac{-0.17047}{0.25470} \right) = 0.81928$$

$\bar{x}_2 = 0.55$	0.45	0.35	0.25	0.15	0.20
		0.24750	0.32863	0.31824	0.25753
$\bar{x}_2 = 0.55$	0.45	0.59750	0.57863	0.46824	0.05753
		0.24750	0.46475	0.57386	
	0.45	0.84500	1.04338	1.04210	

$$p(\bar{x}_2) = 0.05753 \text{ and } p'(\bar{x}_2) = 1.04210, \text{ hence } \bar{x}_{12} = 0.55 - \left(\frac{0.05753}{1.04210} \right) = 0.49479$$

$\bar{x}_2 = 0.85$	0.45	0.35	0.25	0.15	0.20
		0.38250	0.62263	0.74173	0.75797
$\bar{x}_2 = 0.85$	0.45	0.73250	0.87263	0.89173	0.55797
		0.38250	0.94775	1.54732	
	0.45	1.11500	1.82038	2.43905	

$$p(\bar{x}_3) = 0.55797 \text{ and } p'(\bar{x}_3) = 2.43905, \text{ hence } \bar{x}_{13} = 0.85 - \left(\frac{0.55797}{2.43905} \right) = 0.62123.$$

Iterate the procedure for the lower membership function $\underline{x}$.

Step 6: Based on Step 5, arrange the values as follows:

$$\begin{array}{ll} \bar{x}_{11} = 0.81928 & \underline{x}_{11} = 0.41322 \\ \bar{x}_{12} = 0.49479 & \text{and } \underline{x}_{12} = 0.39796 \\ \bar{x}_{13} = 0.62123 & \underline{x}_{13} = 0.57276 \end{array}$$

Step 7: Calculate $\bar{h}_1, \bar{h}_2, \bar{h}_3$ and $\underline{h}_1, \underline{h}_2, \underline{h}_3$ as below:

$$\begin{array}{ll} \bar{h}_1 = 0.81928 - 0.62123 = 0.19805 & \underline{h}_1 = -0.15953 \\ \bar{h}_2 = 0.49479 - 0.62123 = -0.12644 & \text{and } \underline{h}_2 = -0.17480 \\ \bar{h}_3 = (0.19805)(-0.12644)(0.19805 - (-0.12644)) = -0.00812585 & \underline{h}_3 = 0.00042559 \end{array}$$

Step 8: Compute as follows:

$$\begin{array}{l} p(\bar{x}_{11}) = p(0.81928) \rightarrow (0.45)(0.81928)^4 + (0.35)(0.81928)^3 + (0.25)(0.81928)^2 + (0.15)(0.81928) - 0.2 = 0.485912 \\ p(\bar{x}_{12}) = p(0.49479) \rightarrow (0.45)(0.49479)^4 + (0.35)(0.49479)^3 + (0.25)(0.49479)^2 + (0.15)(0.49479) - 0.2 = 0.004791 \\ p(\bar{x}_{13}) = p(0.62123) \rightarrow (0.45)(0.62123)^4 + (0.35)(0.62123)^3 + (0.25)(0.62123)^2 + (0.15)(0.62123) - 0.2 = 0.140606 \\ p(\underline{x}_{11}) = 0.053747 \\ p(\underline{x}_{12}) = 0.039906 \\ p(\underline{x}_{13}) = 0.241899 \end{array}$$

Step 9: Compute as below:

$$\begin{aligned}\bar{A} &= \frac{(-0.12644)(0.485912 - 0.140606) - (0.19805)(0.004791 - 0.140606)}{-0.0081259} = 2.06302 \\ \bar{B} &= \frac{(0.19805)^2 (0.004791 - 0.140606) - (-0.12644)^2 (0.485912 - 0.140606)}{-0.0081259} = 1.33498 \\ \bar{C} &= 0.140606 \\ \underline{A} &= 1.558564 \\ \underline{B} &= 1.42803 \\ \underline{C} &= 0.2419\end{aligned}$$

Step 10: Find the value of $\bar{D}$ and $\underline{D}$ as below:

$$\begin{aligned}\bar{D} &= \sqrt{(1.33498)^2 - 4(0.004791)(0.140606)} \\ &= 0.788586 \\ \underline{D} &= 0.72884\end{aligned}$$

Step 11: Find

$$\begin{aligned}\bar{x}_{new} &= 0.62123 - \frac{2(0.140606)}{1.33498 + 0.788586} = 0.99905 \\ \underline{x}_{new} &= 0.99769\end{aligned}$$

Step 12: Iterate Step 7 to Step 11 until the best approximate solution is obtained.

Table 1 shows the approximate solutions in the form of the upper membership function and the lower membership function.

Table 1. The Approximate Solutions

Upper Membership Function			Lower Membership Function		
t	$\bar{x}$	e	t	$\underline{x}$	e
0	0.9990530426	0.0009469574	0	0.9976876869	0.0023123131
1	1.000000691	-0.000000691	1	1.000002575	-0.000002575
2	1.000000000	0	2	1.000000000	0

Based on the following Table 1:

- The upper membership function of IT2 FPP produced an approximate solution, $\bar{x} = 1.000000000$ after 2 iterations, t and errors, $e = 0$.
- The approximate solutions for lower membership functions are also obtained after 2 iterations, which is $\underline{x} = 1.000000000$ and also has $e = 0$.

The result shows that it has the lowest number of iterations to converge to the approximate solution. Therefore, it can be concluded that Horner-Muller's method successfully produced an approximate solution to solve IT2 DFPP.

Interval Type-2 Fuzzy Pentagonal Polynomial Systems

This section defines the system of interval type-2 fuzzy pentagonal polynomials. The IT2 FPPS can be defined as a combination of the fuzzy polynomial system and interval type-2 pentagonal fuzzy numbers. We presented the IT2 FPPS and performed Horner-Muller's method to obtain the approximate solution of the IT2 FPPS equation.

Definition 2.7: System of fuzzy polynomial (Noor'ani *et al.*, 2010), the equation

$$\begin{cases} A_{11}x + A_{12}x^2 + \dots + A_{1n}x^n = A_{10} \\ A_{21}x + A_{22}x^2 + \dots + A_{2n}x^n = A_{20} \end{cases}$$

for any $x \in R$. The coefficient $A_1', A_2', A_n', A_1'', A_2'', A_n''$ and A_0 are interval type-2 fuzzy pentagonal number.

The IT2 FPPS is defined as follows:

The equation in definition can also be written as follows:

$$\begin{cases} (A_{11}^U, A_{11}^L)x + (A_{12}^U, A_{12}^L)x^2 + \dots + (A_{1n}^U, A_{1n}^L)x^n = (A_{10}^U, A_{10}^L) \\ (A_{21}^U, A_{21}^L)x + (A_{22}^U, A_{22}^L)x^2 + \dots + (A_{2n}^U, A_{2n}^L)x^n = (A_{20}^U, A_{20}^L) \end{cases}$$

Therefore,

$$\begin{cases} \left\{ (\bar{a}_{11}, \bar{a}_{11}, \bar{a}_{11}, \bar{a}_{11}, \bar{a}_{11}), (\underline{a}_{11}, \underline{a}_{11}, \underline{a}_{11}, \underline{a}_{11}, \underline{a}_{11}) \right\} x + \left\{ (\bar{a}_{12}, \bar{a}_{12}, \bar{a}_{12}, \bar{a}_{12}, \bar{a}_{12}), (\underline{a}_{12}, \underline{a}_{12}, \underline{a}_{12}, \underline{a}_{12}, \underline{a}_{12}) \right\} x^2 + \dots + \\ \left\{ (\bar{a}_{1n}, \bar{a}_{1n}, \bar{a}_{1n}, \bar{a}_{1n}, \bar{a}_{1n}), (\underline{a}_{1n}, \underline{a}_{1n}, \underline{a}_{1n}, \underline{a}_{1n}, \underline{a}_{1n}) \right\} x^n = \left\{ (\bar{a}_{10}, \bar{a}_{10}, \bar{a}_{10}, \bar{a}_{10}, \bar{a}_{10}), (\underline{a}_{10}, \underline{a}_{10}, \underline{a}_{10}, \underline{a}_{10}, \underline{a}_{10}) \right\} \\ \left\{ (\bar{a}_{21}, \bar{a}_{21}, \bar{a}_{21}, \bar{a}_{21}, \bar{a}_{21}), (\underline{a}_{21}, \underline{a}_{21}, \underline{a}_{21}, \underline{a}_{21}, \underline{a}_{21}) \right\} x + \left\{ (\bar{a}_{22}, \bar{a}_{22}, \bar{a}_{22}, \bar{a}_{22}, \bar{a}_{22}), (\underline{a}_{22}, \underline{a}_{22}, \underline{a}_{22}, \underline{a}_{22}, \underline{a}_{22}) \right\} x^2 + \dots + \\ \left\{ (\bar{a}_{2n}, \bar{a}_{2n}, \bar{a}_{2n}, \bar{a}_{2n}, \bar{a}_{2n}), (\underline{a}_{2n}, \underline{a}_{2n}, \underline{a}_{2n}, \underline{a}_{2n}, \underline{a}_{2n}) \right\} x^n = \left\{ (\bar{a}_{20}, \bar{a}_{20}, \bar{a}_{20}, \bar{a}_{20}, \bar{a}_{20}), (\underline{a}_{20}, \underline{a}_{20}, \underline{a}_{20}, \underline{a}_{20}, \underline{a}_{20}) \right\} \end{cases}$$

We illustrate numerical examples for considering IT2 FPPS using the Horner-Muller method based on Figure 3 to find its approximate solution.

Example 2: Let we consider the IT2 FPPS as follows:

$$\left\{ \left\{ (0, 0.1, 0.2, 0.3, 0.4), (0, 0.15, 0.2, 0.25, 0.3) \right\} x + \left\{ (0.1, 0.12, 0.13, 0.14, 0.15), (0.24, 0.25, 0.26, 0.27, 0.28) \right\} x^2 + \left\{ (0.2, 0.25, 0.3, 0.35, 0.45), (0.3, 0.32, 0.34, 0.35, 0.4) \right\} x^3 = \left\{ (0.3, 0.47, 0.63, 0.79, 1.0), (0.54, 0.72, 0.8, 0.87, 0.98) \right\}$$

$$\left\{ \left\{ (0, 0.12, 0.22, 0.32, 0.42), (0.1, 0.2, 0.3, 0.4, 0.5) \right\} x + \left\{ (0.14, 0.18, 0.24, 0.28, 0.32), (0.11, 0.12, 0.13, 0.14, 0.15) \right\} x^2 + \left\{ (0.16, 0.18, 0.2, 0.24, 0.26), (0.1, 0.12, 0.14, 0.16, 0.18) \right\} x^3 = \left\{ (0.3, 0.48, 0.66, 0.84, 1), (0.31, 0.44, 0.57, 0.7, 0.83) \right\}$$

$$\left\{ \left\{ (0.15, 0.16, 0.17, 0.18, 0.19), (0.12, 0.14, 0.22, 0.24, 0.32) \right\} x + \left\{ (0.22, 0.23, 0.24, 0.25, 0.26), (0.22, 0.23, 0.24, 0.25, 0.26) \right\} x^2 + \left\{ (0.35, 0.4, 0.45, 0.5, 0.55), (0.1, 0.2, 0.2, 0.3, 0.3) \right\} x^3 = \left\{ (0.72, 0.79, 0.86, 0.93, 1), (0.44, 0.57, 0.66, 0.79, 0.88) \right\}$$

We want to find its approximate solution using the Horner-Muller's method.

Step 1: The parametric form for the upper membership functions are as follows:

$\left[\bar{A}_1 \right]_l^\alpha = 0 - 0.1 + (0.2)(0.5) = 0$	$\left[\bar{A}_1 \right]_U^\alpha = 0.3 - 0.4 + (0.5)(0.5) = 0.15$
$\left[\bar{A}_2 \right]_l^\alpha = 0.1 - 0.12 + (0.13)(0.5) = 0.045$	$\left[\bar{A}_2 \right]_U^\alpha = 0.13 - 0.14 + (0.15)(0.5) = 0.2$
$\left[\bar{A}_3 \right]_l^\alpha = 0.2 - 0.25 + (0.3)(0.5) = 0.1$	$\left[\bar{A}_3 \right]_U^\alpha = 0.3 - 0.35 + (0.45)(0.5) = 0.25$
$\left[\bar{A}_0 \right]_l^\alpha = 0.3 - 0.47 + (0.63)(0.5) = 0.145$	$\left[\bar{A}_0 \right]_U^\alpha = 0.63 - 0.79 + (1)(0.5) = 0.3$
$\left[\bar{A}_1 \right]_l^\alpha = 0 - 0.12 + (0.22)(0.5) = -0.01$	$\left[\bar{A}_1 \right]_U^\alpha = 0.22 - 0.32 + (0.42)(0.5) = 0.15$
$\left[\bar{A}_2 \right]_l^\alpha = 0.14 - 0.18 + (0.24)(0.5) = 0.08$	$\left[\bar{A}_2 \right]_U^\alpha = 0.24 - 0.28 + (0.32)(0.5) = 0.2$
$\left[\bar{A}_3 \right]_l^\alpha = 0.16 - 0.18 + (0.2)(0.5) = 0.08$	$\left[\bar{A}_3 \right]_U^\alpha = 0.2 - 0.24 + (0.26)(0.5) = 0.25$
$\left[\bar{A}_0 \right]_l^\alpha = 0.3 - 0.48 + (0.66)(0.5) = 0.15$	$\left[\bar{A}_0 \right]_U^\alpha = 0.48 - 0.66 + (0.84)(0.5) = 0.3$
$\left[\bar{A}_1 \right]_l^\alpha = 0.15 - 0.16 + (0.17)(0.5) = 0.075$	$\left[\bar{A}_1 \right]_U^\alpha = 0.17 - 0.18 + (0.19)(0.5) = 0.15$
$\left[\bar{A}_2 \right]_l^\alpha = 0.22 - 0.23 + (0.24)(0.5) = 0.11$	$\left[\bar{A}_2 \right]_U^\alpha = 0.24 - 0.25 + (0.26)(0.5) = 0.2$
$\left[\bar{A}_3 \right]_l^\alpha = 0.35 - 0.4 + (0.45)(0.5) = 0.175$	$\left[\bar{A}_3 \right]_U^\alpha = 0.45 - 0.5 + (0.55)(0.5) = 0.25$
$\left[\bar{A}_0 \right]_l^\alpha = 0.72 - 0.79 + (0.86)(0.5) = 0.36$	$\left[\bar{A}_0 \right]_U^\alpha = 0.86 - 0.93 + (1)(0.5) = 0.3$

and the parametric form for the lower membership functions are as follows:

$$\begin{array}{ll} \left[\underline{A}_1 \right]_{\alpha}^{\alpha} = -0.05 & \left[\underline{A}_1 \right]_U^{\alpha} = 0.1 \\ \left[\underline{A}_2 \right]_{\alpha}^{\alpha} = 0.12 & \left[\underline{A}_2 \right]_U^{\alpha} = 0.13 \\ \left[\underline{A}_3 \right]_{\alpha}^{\alpha} = 0.15 & \left[\underline{A}_3 \right]_U^{\alpha} = 0.19 \\ \left[\underline{A}_0 \right]_{\alpha}^{\alpha} = 0.22 & \left[\underline{A}_0 \right]_U^{\alpha} = 0.42 \\ \\ \left[\underline{A}_1 \right]_{\alpha}^{\alpha} = 0.05 & \left[\underline{A}_1 \right]_U^{\alpha} = 0.15 \\ \left[\underline{A}_2 \right]_{\alpha}^{\alpha} = 0.055 & \left[\underline{A}_2 \right]_U^{\alpha} = 0.065 \\ \left[\underline{A}_3 \right]_{\alpha}^{\alpha} = 0.05 & \left[\underline{A}_3 \right]_U^{\alpha} = 0.07 \\ \left[\underline{A}_0 \right]_{\alpha}^{\alpha} = 0.155 & \left[\underline{A}_0 \right]_U^{\alpha} = 0.285 \\ \\ \left[\underline{A}_1 \right]_{\alpha}^{\alpha} = 0.09 & \left[\underline{A}_1 \right]_U^{\alpha} = 0.14 \\ \left[\underline{A}_2 \right]_{\alpha}^{\alpha} = 0.11 & \left[\underline{A}_2 \right]_U^{\alpha} = 0.12 \\ \left[\underline{A}_3 \right]_{\alpha}^{\alpha} = 0 & \left[\underline{A}_3 \right]_U^{\alpha} = 0.05 \\ \left[\underline{A}_0 \right]_{\alpha}^{\alpha} = 0.2 & \left[\underline{A}_0 \right]_U^{\alpha} = 0.31 \end{array}$$

Step 2: Divide the parametric forms of upper membership functions and lower membership functions by 2 as below:

$$\begin{aligned} \bar{a}_1 &= \frac{0 + 0.1 + (-0.01) + 0.11 + 0.075 + 0.085}{2} = 0.18, \\ \bar{a}_2 &= \frac{0.045 + 0.065 + 0.08 + 0.12 + 0.11 + 0.12}{2} = 0.27, \\ \bar{a}_3 &= \frac{0.1 + 0.175 + 0.225 + 0.15 + 0.19}{2} = 0.4225, \\ \bar{a}_0 &= \frac{0.145 + 0.34 + 0.15 + 0.32 + 0.36 + 0.43}{2} = 0.8725 \end{aligned}$$

and

$$\underline{a}_1 = 0.24, \underline{a}_2 = 0.3, \underline{a}_3 = 0.255, \underline{a}_0 = 0.795$$

Step 3: Therefore, the interval type-2 fuzzy pentagonal polynomials is defined as follows:

$$p(\bar{x}) = 0.4225x^3 + 0.27x^2 + 0.18x - 0.8725 \quad \text{and} \quad p(\underline{x}) = 0.255x^3 + 0.300x^2 + 0.24x - 0.795$$

Step 4: We choose three initial approximations as below:

$$\bar{x}_1 = \underline{x}_1 = 0.15, \bar{x}_2 = \underline{x}_2 = 0.55, \bar{x}_3 = \underline{x}_3 = 0.85$$

Step 5: For upper membership function, substitute one by one initial values $\bar{x}_{11}$, $\bar{x}_{12}$ and $\bar{x}_{13}$ into the following arrangements as below:

$\bar{x}_1 = 0.15$	0.42250	0.27000	0.18000	-0.87250
		0.06338	0.05001	0.03450
$\bar{x}_1 = 0.15$	0.42250	0.33338	0.23001	-0.83800
		0.06338	0.05951	
	0.42250	0.39675	0.28952	

$$p(\bar{x}_1) = -0.83800 \text{ and } p'(\bar{x}_1) = 0.28952, \text{ hence } \bar{x}_{11} = 0.15 - \left(\frac{-0.83800}{0.28952} \right) = 3.04446$$

$\bar{x}_2 = 0.55$	0.42250	0.27000	0.18000	-0.87250
		0.23238	0.27631	0.25097
$\bar{x}_2 = 0.55$	0.42250	0.50238	0.45631	-0.62153
		0.23238	0.40411	
	0.42250	0.73475	0.86042	

$$p(\bar{x}_2) = -0.62153 \text{ and } p'(\bar{x}_2) = 0.86042, \text{ hence } \bar{x}_{12} = 0.55 - \left(\frac{-0.62153}{0.86042} \right) = 1.27236$$

$\bar{x}_3 = 0.85$	0.42250	0.27000	0.18000	-0.87250
		0.35913	0.53476	0.60754
$\bar{x}_3 = 0.85$	0.42250	0.62913	0.71476	-0.26496
		0.35913	0.84001	
	0.42250	0.98825	1.55477	

$$p(\bar{x}_3) = -0.26496 \text{ and } p'(\bar{x}_3) = 1.55477, \text{ hence } \bar{x}_{13} = 0.85 - \left(\frac{-0.26496}{1.55477} \right) = 1.02042.$$

Repeating same procedure is carried out for lower membership function $\underline{x}$ with substitute initial values.

Step 6: Based on Step 5, arrange the values as follows:

$$\begin{array}{ll} \bar{x}_{11} = 3.04446 & \underline{x}_{11} = 2.31406 \\ \bar{x}_{12} = 1.27236 & \text{and } \underline{x}_{12} = 1.21111 \\ \bar{x}_{13} = 1.02042 & \underline{x}_{13} = 1.01707 \end{array}$$

Step 7: Calculate $\bar{h}_1, \bar{h}_2, \bar{h}_3$ and $\underline{h}_1, \underline{h}_2, \underline{h}_3$ as below:

$$\begin{array}{ll} \bar{h}_1 = 2.22593 - 1.01788 = 2.02404 & \underline{h}_1 = 1.82208 \\ \bar{h}_2 = 1.21880 - 1.01788 = 0.25194 & \text{and } \underline{h}_2 = 0.24334 \\ \bar{h}_3 = (1.06096)(0.18636)(1.06096 - (0.18636)) = 0.90366587155 & \underline{h}_3 = 0.69997 \end{array}$$

Step 8: Compute as follows:

$$p(\bar{x}_{11}) = p(2.22593) \rightarrow (0.275)(2.22593)^3 + (0.2)(2.22593)^2 + (0.25)(2.22593) - 0.725 = 3.85540$$

$$p(\bar{x}_{12}) = p(1.21880) \rightarrow (0.275)(1.21880)^3 + (0.2)(1.21880)^2 + (0.25)(1.21880) - 0.725 = 0.37468$$

$$p(\bar{x}_{13}) = p(1.01788) \rightarrow (0.275)(1.01788)^3 + (0.2)(1.01788)^2 + (0.25)(1.01788) - 0.725 = 0.02670$$

$$\underline{p}(\underline{x}_{11}) = 10.92112$$

$$\underline{p}(\underline{x}_{12}) = 0.60834$$

$$\underline{p}(\underline{x}_{13}) = 0.03870$$

Step 9: Compute as below:

$$\bar{A} = \frac{(0.20092)(3.85540 - 0.02670) - (1.20805)(0.37468 - 0.02670)}{0.24445} = 1.42722$$

$$\bar{B} = \frac{(1.20805)(0.37468 - 0.02670) - (0.20092)^2(3.85540 - 0.02670)}{0.24445} = 1.44517$$

$$\bar{C} = 0.02670$$

$$\underline{A} = 2.30024$$

$$\underline{B} = 1.78131$$

$$\underline{C} = 0.03870$$

Step 10: Find the value of $\bar{D}$ and $\underline{D}$ as below:

$$\begin{aligned}\bar{D} &= \sqrt{(1.44517)^2 - 4(1.42722)(0.02670)} \\ &= 1.39143\end{aligned}$$

$$\underline{D} = 1.67839$$

Step 11: Find

$$\bar{x}_{new} = 1.017880 - \frac{2(0.02670)}{1.44517 + 1.39143} = 0.99905$$

$$\underline{x}_{new} = 0.99769$$

Step 12: Iterate Step 7 to Step 11 until the best approximate solution is obtained.

Table 2 shows the approximate solutions in the form of the upper membership function and the lower membership function.

Table 2: The Approximate Solutions

Upper Membership Function			Lower Membership Function		
t	$\bar{x}$	e	t	$\underline{x}$	e
0	0.9972154194	0.0009469574	0	0.9992089387	0.0023123131
1	1.000003295	-0.000000691	1	1.000000453	-0.000002575
2	1.000000000	0	2	1.000000000	0

Based on the following Table 2:

- Upper membership function of IT2 FPPS produced an approximate solution, $\bar{x} = 1.000000000$ after 2 iterations, t and errors, $e = 0$.
- The approximate solutions for lower membership functions are also obtained after 2 iterations, which is $\bar{x} = 1.000000000$ and also has $e = 0$.

The result shows that it requires the fewest iterations to converge to the approximate solution. Therefore, it can be concluded that IT2 FPPS successfully solves an approximate solution using Horner-Muller's method.

Conclusions

This study introduced novel formulations for interval type-2 fuzzy pentagonal polynomial systems and interval type-2 dual fuzzy pentagonal polynomial equations. By integrating interval type-2 pentagonal fuzzy numbers, these formulations enhance the existing fuzzy polynomial framework by capturing uncertainty and imprecision in the coefficients of polynomials. The suggested approach offers a mathematical framework to analyze dual fuzzy polynomial equations and fuzzy polynomial systems within a pentagonal fuzzy context, specifically in the context of interval type-2.

To obtain approximate solutions for the suggested interval type-2 fuzzy pentagonal polynomial formulations, the iterative Horner-Muller method was employed. The computational methods using interval type-2 dual fuzzy pentagonal polynomial equations and interval type-2 fuzzy pentagonal polynomial systems were illustrated by the numerical examples. In conclusion, this research broadens the application of interval type-2 pentagonal fuzzy numbers to dual fuzzy polynomial equations and polynomial systems by demonstrating the application of the suggested formulations and the method of obtaining their approximate solutions. Additional research into more intricate fuzzy polynomial issues that encompass greater degrees of uncertainty is made possible by the suggested formulations. This study proposed new formulations for interval type-2 dual fuzzy pentagonal polynomial equations and interval type-2 fuzzy pentagonal polynomial systems. Future studies might explore expanding the suggested method to higher-degree polynomial systems, examining different numerical solution techniques, and assessing its relevance to real-world problems that involve uncertain and imprecise data. By incorporating interval type-2 pentagonal fuzzy numbers to represent uncertainty and imprecision in polynomial coefficients, these formulations extend the existing fuzzy polynomial framework. Thus, the proposed method offers a mathematical framework for examining fuzzy polynomial equations and dual fuzzy polynomial equations.

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